LEBANESE UNIVERSITY
FACULTY OF ENGINEERING

الجامعة اللبنانية
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Entrance Exam 2006 -2007 **Mathematics** **Duration : 3 hours**

The distribution of grades is over 25

I-(2 pts) The adjacent table is that of a continuous function f defined on IR. The représentative curve (C) of f admits at $+\infty$ an asymptotic direction parallel to the straight line of equation $y = x$.

x	$-\infty$		1		$+\infty$
$f''(x)$		+		+	
$f'(x)$		+	½	+	
$f(x)$	0		1		$+\infty$

1-Determine an equation of the tangent (T) to (C) at the point of abscissa 1.
2- Draw (C) and (T).
3- Prove that, for all $x \in]1 ; +\infty[$, $(x+1)/2 < f(x) < x$.

II-(4 pts) An urn contains 6 identical balls of which 4 are red and 2 are black.
1- We randomly draw two balls from the urn. Consider the three events:
A_0 : the two drawn balls are red
A_1 : the two drawn balls have different colors.
A_2 : the two drawn balls are black.
Calculate the probability of each of A_0, A_1 and A_2.

2- After the first drawing, the urn contains 4 balls. We randomly draw two new balls from the urn. Consider the three events:
B_0 : the two drawn balls are red
B_1 : the two drawn balls have different colors.
B_2 : the two drawn balls are black.

a) Calculate $p(B_0 / A_0)$, $p(B_0 / A_1)$ and $p(B_0 / A_2)$. Deduce that $p(B_0) = 0.4$
b) Calculate $p(B_1)$ and $p(B_2)$
c) Knowing that only one black ball is obtained in the second drawing, calculate the probability that only one black ball has been obtained in the first drawing.

3- Calculate the probability that, after the two drawing, the remaining two balls in the urn are red.

III- (6 pts) The space is referred to a direct orthonormal system $(O ; \vec{i}, \vec{j}, \vec{k})$
Consider the point $A(-1 ; 1 ; 0)$, the plane (P) of equation $x - 2y + 2z - 6 = 0$ and the straight line (D) defined by the system $x = m+1$; $y = 2m+1$; $z = 3m+2$.

1- Prove that A and (D) determine a plane (Q) and determine an equation of (Q).
2- a) Prove that (P) and (Q) intersect along the straight line (Δ) defined by $x = 2$; $y = t - 2$; $z = t$.
b) Determine the coordinates of A', the orthogonal projection of A on (Δ).
3- M is a variable point of (Δ). Let α be a measure of the angle that (AM) makes with (P).
a) Prove that $AM \times \sin \alpha = 3$
b) Determine the position of M so that α is maximum. Calculate $\sin \alpha$ in this case.
c) What does this value of α represent for the two planes (P) and (Q)?

4- Consider the circle (C) of center A tangent to (Δ) and lying in the plane (Q). The orthogonal projection of (C) on the plane (P) is an ellipse (E).
  a) Calculate the radius of (C).
  b) Determine the coordinates of the center of (E).
  c) Calculate the eccentricity of (E).
  d) Determine a system of parametric equation of the focal axis of (E).
  e) Determine the coordinates of each of the two foci of (E).
  f) Calculate the area of the domain bounded by (E) and its auxiliary circle (γ) .

IV- (6 pts) The complex plane is referred to a direct orthonormal system $(O\,;\,\vec{u}\,,\,\vec{v})$
Consider the points A_0, A_1 and A_2 of respective affixes $z_0 = 5-4i$, $z_1 = -1-4i$ et $z_2 = -4-i$. Let S be the similitude that transforms A_0 into A_1 and A_1 into A_2
1-a) Determine the ratio of S.
  b) Prove that the point I (2 ; 2) is the center of the circle (γ) circumscribed about the triangle $A_0A_1A_2$.
  c) Calculate the radius of the image of (γ) by (S).
2-a) Prove that the complex expression of S is $z' = \frac{1-i}{2}z + \frac{i-3}{2}$.
  b) Deduce the angle of S and the affix d of its center D.
3-Let M be any point of affix z, such that $z \neq d$, and M', with affix z', its image by S.
  a) Determine the nature of the triangle DMM'
  b) Deduce that $d - z' = i(z - z')$.
4-Consider the sequence of points (A_n) of first term A_0 defined by $A_{n+1} = S(A_n)$
  Let (U_n) be the sequence defined on IN by $U_n = A_n A_{n+1}$
  a) Plot the points A_0, A_1, A_2 and construct the points A_3, A_4, A_5
  b) Prove that the sequence (U_n) is geometric.
5- Consider the sequence of points (P_k) defined by $P_k = A_{m+4k}$ where m is a <u>given</u> natural number
  a) Prove that all the points P_k are collinear.
  b) Prove that, for all natural numbers k, $P_{k+1} = H(P_k)$ where H is a transformation to be determined.

V-(7 pts) **<u>A.</u>** Consider the differential equation (1) $y' + 2y^2e^x - y = 0$ where y is a function defined on IR, such that, for all x in IR, $y(x) \neq 0$. Let $z = \frac{1}{y}$ where z is a differentiable function defined on IR.

1- Determine the differential equation (2) satisfied by z.
2- Solve the equation (2) and deduce the general solution of equation (1).
3- Determine the particular solution of equation (1) that satisfies the condition $y(0) = \frac{1}{2}$

B. Let f be the function defined on IR by $f(x)=\dfrac{1}{e^x+e^{-x}}$. Designate by (C) the representative curve of f in an orthonormal system $(O\,;\,\vec{i}\,,\,\vec{j}\,)$

1- Prove that f is an even function.
2- Set up the table of variations of f .
3-a) Set up the table of variations of the function g defined on $[0\,;+\infty[$ by $g(x)=f(x)-x$.
b) Deduce that the equation $f(x)=x$ admits in $[0\,;+\infty[$ only one solution α. Verify that $0{,}4<\alpha<0{,}5$.
c) Draw (C) **(Unit = 4 cm).**
4-a) Prove that the restriction de f to the interval $[0\,;+\infty[$ admits an inverse function f^{-1}.
b) Determine the domain of definition of f^{-1} and calculate $f^{-1}(x)$.
c) Draw the curve (γ) of f^{-1} in the same system as (C).

C. Let (V_n) be the sequence defined on IN by $V_n=\displaystyle\int_0^n f(x)dx$

1-a) Prove that, for all $x\geq 0$, $f(x)<e^{-x}$
b) Deduce that, for all n in IN, $V_n\leq 1-e^{-n}$

2-a) Verify that $V_{n+1}-V_n=\displaystyle\int_n^{n+1} f(x)dx$
b) Deduce that the sequence (V_n) is strictly increasing.
c) Prove that the sequence (V_n) is convergent to a limit ℓ such that $0\leq\ell\leq 1$.

3-Verify that $f(x)=\dfrac{e^x}{1+e^{2x}}$ then Calculate V_n in terms of n and determine ℓ

4- Calculate in cm^2 the area of the domain bounded by (γ), $y'y$, $x'x$ and the straight line of equation y = 2.

Entrance Exam 2006 -2007 **Solution of Mathematics** **Duration:3 hours**

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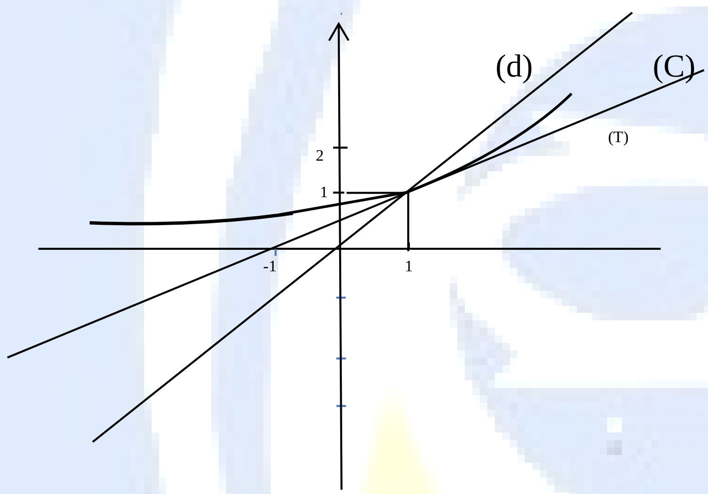
Exercice I

1- The equation of the tangent (T) to (C) at the point of abscissa 1is y = f(1) + f ´(1) (x-1) =1+$\frac{1}{2}$ (x-1)

$f'(1) = \frac{1}{2}$ and $f(1) = 1$ so $y = \frac{1}{2}x + \frac{1}{2} = \frac{1}{2}(x+1)$

2-



x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$	0	$\nearrow$	$+\infty$

3) For x >1, the representative curve of f is above the tangent and below the straight line of equation y = x hence $\frac{(x+1)}{2} < f(x) < x$

Exercise II

1-p(A$_0$)= $\frac{C_4^2}{C_6^2}=\frac{2}{5}$

p(A$_1$)= $\frac{C_4^1C_2^1}{C_6^2}=\frac{8}{15}$

p(A$_2$)= $\frac{C_2^2}{C_6^2}=\frac{1}{15}$

2-a) p(B$_0$ / A$_0$)= $\frac{C_2^2}{C_4^2}=\frac{1}{6}$, p(B$_0$ / A$_1$)= $\frac{C_3^2}{C_4^2}=\frac{1}{2}$, p(B$_0$ / A$_2$)= 1

p(B$_0$) = p(B$_0$ / A$_0$).p(A$_0$) + p(B$_0$ / A$_1$).p(A$_1$)+ p(B$_0$ / A$_2$).p(A$_2$)= $\frac{1}{6}.\frac{2}{5}+\frac{1}{2}.\frac{8}{15}+\frac{1}{15}=\frac{2}{5}=0.4$

b) p(B$_1$) = p(B$_1$ / A$_0$).p(A$_0$) + p(B$_1$ / A$_1$).p(A$_1$)+ p(B$_1$ / A$_2$).p(A$_2$)= $\frac{2}{3}.\frac{2}{5}+\frac{1}{2}.\frac{8}{15}+0=\frac{8}{15}$

or p(B$_1$ / A$_2$) =0

p(B$_2$)= $\frac{1}{6}.\frac{2}{5}=\frac{1}{15}$

c) p(A$_1$/B$_1$) = $\frac{p(A_1\cap B_1)}{p(B_1)}=\frac{p(B_1/A_1)\times p(A_1)}{p(B_1)}=\frac{1}{2}$

3- p(2R)= p(A$_0$) x p(B$_2$ / A$_0$)+p(A$_1$) x p(B$_1$ / A$_1$)+ p(B$_0$ / A$_2$).p(A$_2$)= $\frac{2}{5}.\frac{1}{6}+\frac{1}{2}.\frac{8}{15}+1.\frac{1}{15}=\frac{2}{5}$

Exercise III

1) The point A does not belong to (d) since if -1 = m+1, 1= 2m+1, 0= 3m+2

We get m = -2, m = 0, m = $-\frac{2}{3}$

Hence A and (d) determine a plan (Q). B (1, 1, 2) is a point of (d) and M(x, y, z) is a variable point of (Q). An equation of (Q) is $\overrightarrow{AM}.(\overrightarrow{AB}\wedge\vec{v}_d)=0$ which gives x+ y-z =0

2) a- $\overrightarrow{n_p}(1;-2;2)$ and $\overrightarrow{n_Q}(1;1;-1)$ are not collinear then (P) and (Q) intersect along the straight line (Δ)

Let M (2; t-2; t) be a variable point of (Δ)

M$\in$(P) since $x_M-2y_M+2z_M-6=2-2t+4+2t-6=0$

M$\in$(Q) since $x_M+y_M-z_M=2+t-2-t=0$

Hence x = 2, y = t-2, z = t is a system of parametric equations of (Δ).

b- A' is a point of (Δ) then A' (2; t-2; t), $\overrightarrow{AA'}$(3; t-3; t) and $\vec{v}_\Delta(0;1;1)$ are orthogonal, then $\overrightarrow{AA'}.\vec{v}_\Delta = 0$ which gives t-3+t = 0, then t = $\frac{3}{2}$ and consequently $A'(2;-\frac{1}{2};\frac{3}{2})$

3) a- Let h be the orthogonal projection of A on (P), the angle that (AM) makes with (P) is $\widehat{AMH}$.

$\sin\alpha = \frac{\text{HA}}{\text{AM}}$ but $HA = d(A;P) = \frac{|-1-2-6|}{\sqrt{1+4+4}} = 3$

then $\sin\alpha = \frac{3}{\text{AM}}$ and consequently $AM.\sin\alpha = 3$

b- α is maximum when AM is minimum that is when M is confounded with A' in this case $AA'(3;-\frac{3}{2};\frac{3}{2})$ therefore $AA' = \frac{3}{2}\sqrt{6}$

$$\sin\alpha = \frac{\text{HA}}{\text{AA}'} = \frac{3}{\frac{3}{2}\sqrt{6}} = \frac{\sqrt{6}}{3}$$

c- (AH) $\perp$ (P) then (AH) $\perp$ (Δ) and since (AA') $\perp$ (Δ) then (Δ) $\perp$ (A'H) hence α is the plane angle of the dihedral of (P) and (Q).

4) a- The radius of (C) is r = $AA' = \frac{3}{2}\sqrt{6}$

b- The center of (E) is the point H. The vector r $\overrightarrow{n_p}(1;-2;2)$ is a direction vector of (AH). A system of parametric equations of (AH) is $x = \lambda - 1,\ y = -2\lambda + 1,\ z = 2\lambda$. H is the point of intersection of (AH) and (P), then $\lambda - 1 + 4\lambda - 2 + 4\lambda - 6 = 0$ which gives λ =1 so H(0; -1; 2)

c- We know that a = r, b = rcosα then, $c^2 = a^2 - b^2 = r^2 - r^2\cos^2\alpha = r^2\sin^2\alpha$ so c = rsinα = 3

$$e = \frac{c}{a} = \frac{r\sin\alpha}{5}\sin\alpha = \frac{\sqrt{6}}{3}$$

d- The focal axis of (E) is the straight line passing through the center of (E) and parallel to the line (Δ) then $\vec{v}_{\Delta}(0;1;1)$ is the direction vector of the focal axis, a system of parametric equation of the focal axis is : x = 0, y = k-1, z = k+2

e- Let F be one focus of (E) , F belongs to the focal axis then: F(0; k-1; k+2), HF = c =3 so $HF^2 = 9$.
But $\overrightarrow{HF}(0;k;k)$ so $k^2 + k^2 = 9$ which gives $k=\frac{3\sqrt{2}}{2}$ or $k=-\frac{3\sqrt{2}}{2}$ therefore $F(0;\frac{3\sqrt{2}}{2}-1;\frac{3\sqrt{2}}{2}+2)$
And $F(0;-\frac{3\sqrt{2}}{2}-1;-\frac{3\sqrt{2}}{2}+2)$

f- The area of the auxiliary circle is $S_1=\pi\times a^2=\pi\times r^2$ and the area of the ellipse is $S_2=\pi\, a\, b=\pi\times r\times r\cos\alpha$, then the area of the domain bounded by (E) and its auxiliary circle is:
S = S1 – S2 = $\pi\times r^2-\pi\times r^2\cos\alpha=\pi\times r^2(1-\cos\alpha)$ but $\cos\alpha=\sqrt{1-\sin^2\alpha}=\frac{\sqrt{3}}{3}$ and $r=\frac{3}{2}\sqrt{6}$
then, $S=\frac{27}{2}\pi(1-\frac{\sqrt{3}}{3})$ square units.

Exercise IV

1) a- $k=\frac{A_1A_2}{A_0A_1}=\frac{|z_2-z_1|}{|z_1-z_0|}=\frac{|-3+3i|}{|-6|}=\frac{\sqrt{2}}{2}$

b- $IA_0=|z_0-z_I|=|5-4i-2-2i|=|3-6i|=3\sqrt{5}$, $IA_1=|z_1-z_I|=|-1-4i-2-2i|=|-3-6i|=3\sqrt{5}$,
$IA_2=|z_2-z_I|=|-4-i-2-2i|=|-6-3i|=3\sqrt{5}$
Then $IA_0 = IA_1 = IA_2$ consequently I (2; 2) is the center of circle (γ) circumscribed about triangle $A_0A_1A_2$

c- The image of (γ) by S is a circle of radius, R' = $K.R=\frac{\sqrt{2}}{2}.3\sqrt{5}=\frac{3}{2}\sqrt{10}$

2) a- The complex expression of a similitude is z´= az + b ; S(A_0) = A_1 gives : $z_{A_1}= az_{A_0}+b$ and S(A_1) = A_2 gives : $z_{A_2}= az_{A_1}+b$ we get the system :
(5-4i) a+b= -1- 4i
(-1-4i) a+b= -4-i that has no solution $a=\frac{1}{2}-\frac{1}{2}i$ and $b=-\frac{3}{2}+\frac{1}{2}i$ then $z'=\frac{1-i}{2}z+\frac{i-3}{2}$

b- $a = \left(\dfrac{1-i}{2}\right) = \dfrac{\sqrt{2}}{2}\left(\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}\right) = \dfrac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}}$

then an angle of S is $-\dfrac{\pi}{4}$ the affix d of the center D of S is $d = \dfrac{b}{1-a} = -1+2i$

3) a- $DM' = \dfrac{\sqrt{2}}{2} DM$ and $(\vec{DM};\vec{DM'}) = -\dfrac{\pi}{4}(2\pi)$. Let $DM = \ell$ so $DM' = \dfrac{\sqrt{2}}{2}\ell$ then

$MM'^2 = DM^2 + DM'^2 - 2DM \times DM' \cos(\dfrac{\pi}{4})$

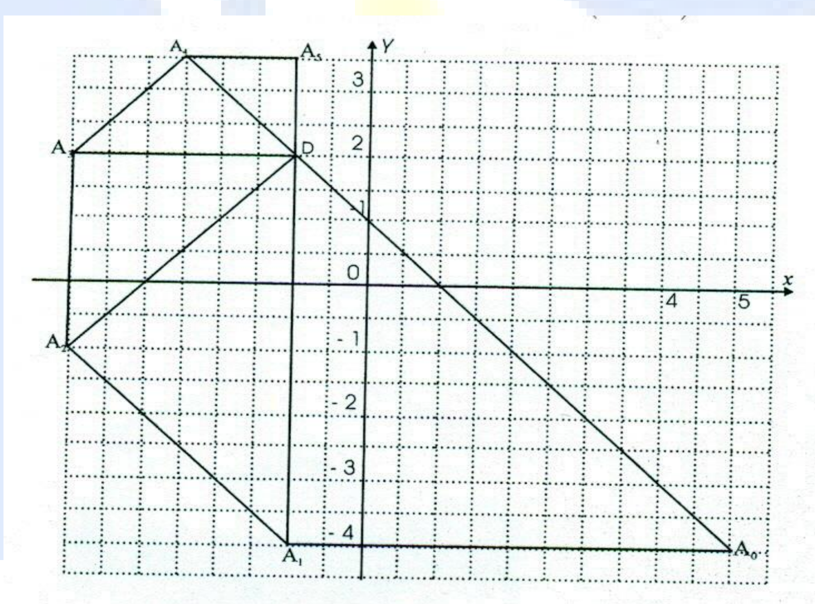
$MM'^2 = \ell^2 + \dfrac{\ell^2}{2} - 2\ell \times \ell \dfrac{\sqrt{2}}{2} \times \dfrac{\sqrt{2}}{2} = \dfrac{\ell^2}{2}$ so $MM' = \dfrac{\sqrt{2}}{2}\ell$ then the triangle DMM' is isosceles at M' and

since $M\hat{D}M' = \dfrac{\pi}{4}$ then DMM' is right isosceles at M'.

b- $\dfrac{z'-d}{z'-z} = \dfrac{DM'}{MM'} e^{i(\vec{MM'};\vec{DM'})} = 1e^{i\frac{\pi}{2}} = i$

$z' - d = i(z' - z)$ and d - z' = i(z - z')

4) a-



b- $\frac{U_{n+1}}{U_n} = \frac{A_{(n+1)}A_{(n+2)}}{A_{(n)}A_{(n+1)}}$ *but* $A_{n+1} = S(A_n)$ *and* $A_{n+2} = S(A_{n+1})$ *then* $\frac{A_{(n+1)}A_{(n+2)}}{A_{(n)}A_{(n+1)}} = \frac{\sqrt{2}}{2}$ and consequently

$\frac{U_{n+1}}{U_n} = \frac{\sqrt{2}}{2} \Rightarrow (U_n)$ is a geometric sequence of common ratio $\frac{\sqrt{2}}{2}$ and of first term $U_0 = A_0A_1 = 6$

5) a- $(\vec{DP_k};\vec{DP_{k+1}}) = (\vec{DA_{m+4k}} + \vec{DA_{m+4k+4}}) = (\overrightarrow{DA_{m+4k}} + \overrightarrow{DA_{m+4k+1}}) + \overrightarrow{DA_{m+4k+1}};\overrightarrow{DA_{m+4k+2}}) + (\overrightarrow{DA_{m+4k+2}};\overrightarrow{DA_{m+4k+3}})$
$+ (\overrightarrow{DA_{m+4k+3}};\overrightarrow{DA_{m+4k+4}}) = -\frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{4} = -\pi(2\pi)$

Then the points D, P_k and P_{k+1} are collinear; consequently all the points P_k belong to the same straight line passing through D.

b- $\frac{DP_{(k+1)}}{DP_{(k)}} = \frac{DA_{(m+4k+4)}}{DA_{(m+4k)}} = \frac{DA_{(m+4k+4)}}{DA_{(m+4k+3)}} \cdot \frac{DA_{(m+4k+3)}}{DA_{(m+4k+2)}} \cdot \frac{DA_{(m+4k+2)}}{DA_{(m+4k+1)}} \cdot \frac{DA_{(m+4k+1)}}{DA_{(m+4k)}}$

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = \frac{1}{4} \Rightarrow (\overrightarrow{DP_k};\overrightarrow{DP_{k+1}}) = -\pi(2\pi)$ *then* $\overrightarrow{DP_{k+1}} = -\frac{1}{4}\overrightarrow{DP_k}$

Consequently $P_{k+1} = H(P_k)$ where h is the dilatation of center D and ratio -1/4

Exercice V

A-1) $z = \frac{1}{y} \Rightarrow$ y = 1/z so

$y' = \frac{z'}{z^2}$ then $y' + 2y^2e^x - y = 0$

Gives $-\frac{z'}{z^2} + 2\frac{1}{z^2}e^x - \frac{1}{z} = 0$ that is $-z' + 2e^x - z = 0$ and $(\beta); z' + z = 2e^x$

2) The general solution of $z' + z = 0$ is $z_1 = C\,e^{-x}$

$z_2 = e^x$ is a particular solution of the equation (β) then
$z = z_1 = z_2 = Ce^{-x} + e^x$ is a general solution of (β)
$y(x) = \frac{1}{z(x)} = \frac{1}{Ce^{-x} + e^x}$ is a general solution of (α)

3) $y(0) = \frac{1}{2} \Rightarrow c = 1 \Rightarrow y(x) = \frac{1}{e^x + e^{-x}}$ is a particular solution of (α)

B- 1) The domain of f is centered at O , and f (-x) = $\frac{1}{e^{x}+e^{-x}} = f(x) \Rightarrow f$ is an even function

2) f´(x)= $\frac{e^{x}-e^{-x}}{(e^{x}+e^{-x})} = \frac{1-e^{2x}}{1+e^{2x}}$, $f'(x) \geq 0$ for $x \leq 0$ than the table of variations of f is :

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} f(x) = 0$

x	-∞	0	+∞
f ´ (x)	+	0	-
f (x)	↗	1/2	↘

3) a- $g'(x) = f'(x)$ -, then for $x > 0$, $f'(x) < 0$ so

$x > 0$ $g'(x) < 0$ than the table of variations of g is

x	0		+∞
g ´ (x)		-	
g (x)	1/2	↘	-∞

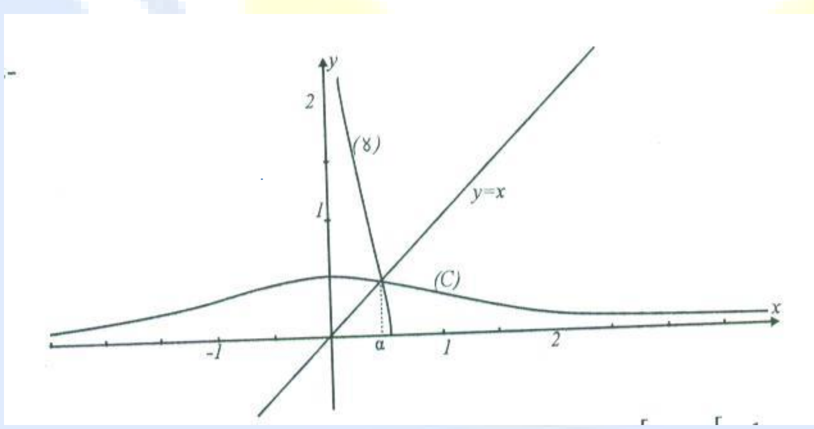
$\lim\limits_{x \to +\infty} g(x) = \lim\limits_{x \to +\infty} [f(x) - x] = 0 - \infty = -\infty$

b- g is continuous and strictly decreasing over [0 ; +∞[, it decrease from $\frac{1}{2}$ to -∞ then its representative curve cuts the axis $x'x$ at a unique point, consequently g(x) = 0 has one root α , so the equation $f(x) = x$ has one solution α over [0 ; +∞[

$g(0{,}4) = f(0{,}4) - 0{,}4 = 0{,}0625 > 0$

$g(0{,}6) = f(0{,}6) - 0{,}6 = -0{,}056 < 0$ therefore $0{,}4 < \alpha < 0{,}5$

c)



4) a- f is continuous and strictly decreasing over [0; +∞ [, then it admits an inverse function f^{-1}

b- The domain of definition of f^{-1} is $\left]0;\frac{1}{2}\right]$

$y = \frac{1}{e^x + e^{-x}} \Rightarrow y = \frac{e^x}{1+e^{2x}}$ then $ye^{2x} + y - e^x = 0$

quadratic equation in e^x, Δ=1-4y^2 ; $e^x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$; which gives

$x = \ln(\frac{1 \pm \sqrt{1-4y^2}}{2y})$ for $y = \frac{1}{4}; x = \ln\left[\frac{1+\sqrt{1-\frac{1}{4}}}{\frac{1}{2}}\right] = \ln(2+\sqrt{3}) > 0$ or $x = \ln\left[\frac{1-\sqrt{1-\frac{1}{4}}}{\frac{1}{2}}\right] = \ln(2-\sqrt{3}) < 0$

so the accepted solution is $x = \ln\left[\frac{1+\sqrt{1-4y^2}}{2y}\right] \Rightarrow f^{-1}(x) = \ln\left[\frac{1+\sqrt{1-4x^2}}{2x}\right]$

c- Drawing of the graph (γ) of f^{-1} in the same system as that of (C).

(v_n) is the sequence defined on IN by $v_n = \int_0^n f(x)dx$

C- 1) a- $f(x) - e^{-x} = \frac{1}{e^x + e^{-x}} - e^{-x} = \frac{1-1-e^{-2x}}{e^x + e^{-x}} = \frac{-e^{-2x}}{e^x + e^{-x}} < 0$ then $f(x) < e^{-x}$

for all x and in particular for $x \geq 0$

b- $f(x) < e^{-x}$ then $\int_0^n f(x)dx < \int_0^b e^{-x}dx$ so $\int_0^n f(x)dx < \left[-e^{-x}\right]_0^n$

$\int_0^n f(x)dx < 1 - e^{-n} \Rightarrow V_n \leq 1 - e^{-n}$

2) a- $V_{n+1} - V_n = \int_0^{n+1} f(x)dx - \int_0^n f(x)dx = \int_n^0 f(x)dx + \int_0^{n+1} f(x)dx = \int_n^{n+1} f(x)dx$

b – Since $f(x) > 0$ then $\int_n^{n+1} f(x)dx > 0$ so $v_{n+1} - v_n > 0$ then $v_{n+1} > v_n$ consequently the sequence (v_n) is strictly increasing.

c – The sequence (v_n) is increasing and bounded above by 1 then $V_n \le 1 - e^{-n} < 1$ so it is convergent to a limit ℓ.
Since $0 \le v_n < 1$ then $0 \le \ell \le 1$

3) $f(x) = \dfrac{1}{e^x + e^{-x}} = \dfrac{e^x}{e^{2x}+1}$

$$v_n = \int_0^n \frac{e^x}{e^{2x}+1}dx = \left[\arctan e^x\right]_0^n = \arctan e^n - \arctan 1 = \arctan e^n - \frac{\pi}{4}$$

$$\lim_{n\to+\infty} v_n = \lim_{n\to\infty} \arctan e^n - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \text{since } \lim_{n\to\infty} \arctan e^n = \arctan(+\infty) = \frac{\pi}{2}$$

4) The required area is $\int_0^2 f(x)dx = v_2 = \arctan e^2 - \dfrac{\pi}{4}$ square units $= 16 \times (\arctan e^2 - \dfrac{\pi}{4})cm^2$