Lebanese University
Faculty of Engineering

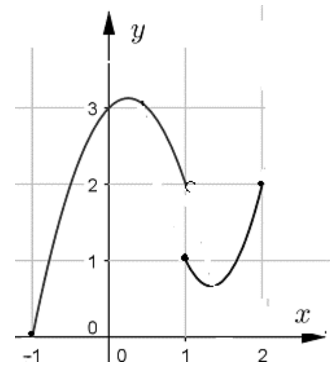
Entrance Exam
MATHEMATICS
TEST A
(Lebanese Program)

Date : *10/07/2024*
Time : *2h30*

Smartphones and notes are strictly prohibited.
Only non graphical calculators are allowed.

***Exercise 1* :** (12 points)

This exercise is a multiple-choice quiz. For each question, ***only one*** *of the four answers given is correct. A wrong answer, a multiple answer or the absence of an answer to a question does not earn or deduct points. For this exercise, you have to answer on the answers sheet, by circling for each question* ***only one*** *of the letters* ***a***, ***b***, ***c*** *or* ***d.***
No justification is required.



1. Consider the function f defined in $[-1\,;2]$, whose curve is shown on the figure.

a) For all real number k, the equation $f(x) = k$ admits exactly 2 solutions.
b) For all real number $k > 3$, the equation $f(x) = k$ admits no solution.
c) The equation $f(x) = x$ admits exactly 2 solutions.
d) None of the previous statements is correct.

2. Consider the sequence (U_n) defined by $U_0 = 1$ and $U_{n+1} = 0{,}9U_n + 20$ for all $n \in \mathbb{N}$.
The sequence (V_n) defined by $V_n = U_n - 200$ is :
a) arithmetic. **b)** geometric. **c)** decreasing. **d)** None of the previous answers is correct.

3. The complex number $z = \left(\frac{1}{\sqrt{2}}(1 - i\sqrt{3})\right)^{18}$ equals:

a) -512 **b)** $\frac{\sqrt{3}}{2} - \frac{1}{2}i$ **c)** 512 **d)** None of the previous answers is correct.

4. Let (U_n) be the sequence defined by $U_n = \left(\frac{2}{3}\right)^n$. The sequence (S_n) defined by $S_n = U_0 + U_1 + \cdots + U_n$ is :
a) decreasing. **b)** convergent. **c)** divergent. **d)** None of the previous answers is correct.

5. Let f be the function defined by $f(x) = \begin{cases} e^{-\frac{1}{x+1}} & if \quad x > -1 \\ \sqrt{|ax+1|} & if \quad x \leq -1 \end{cases}$ where a is a real constant.
a) The function f is continuous if $a = -1$
b) The function f is continuous if $a = 1$
c) The function f is continuous if $a = 0$
d) For all $a \in \mathbb{R}$, the function f is never continuous.

6. Consider the function defined by $f(x) = 5\ln(xe^x)$.
a) The function f is convex.
b) The function f is concave.
c) f is not convex, nor concave.
d) None of the previous statements is correct.

7. The set of points M of affix $z \in \mathbb{C} - \{1\}$ such that $\frac{z+1}{z-1}$ is pure imaginary is :
a) The straight line (Ox) except the point $(1\,,0)$.
b) The straight line (Oy) except the point $(0\,,1)$.
c) The circle of center O and of radius 1.
d) None of the previous answers is correct.

8. A multiple-choice questionnaire (MCQ) is made up of eight questions. For each of them, four answers are proposed, only one of which is correct. A candidate answers all the questions at random.

a) The number of possible answers to this MCQ is 4096.
b) The number of possible answers to this MCQ is 65536.
c) The probability that the candidate will answer exactly 6 questions correctly is 3/4.
d) None of the previous statements is correct.

9. Consider the equation $\ln(x) + \ln(x-1) = \ln(x+1)$, of real unknown x.
The solution set S of this equation is :

a) $S = \{ 2 \}$
b) $S = \{ -1+\sqrt{2} \; ; \; -1-\sqrt{2} \}$
c) $S = \{ 1+\sqrt{2} \; ; 1-\sqrt{2} \}$
d) None of the previous answers is correct.

10. A bottle initially containing 1 liter of water is left in the sun. Every hour, the water volume decreases by 15%.
The volume of water becomes less than a quarter of a liter after:

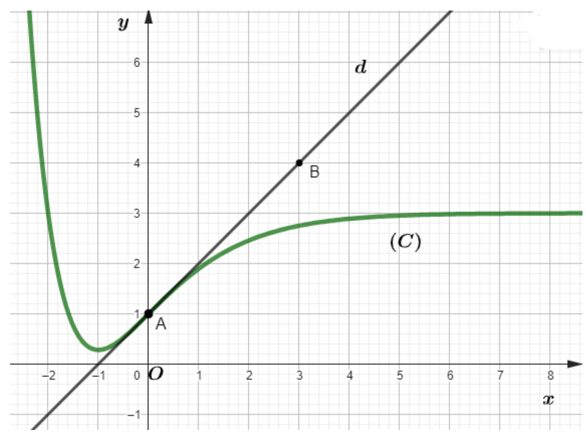
a) 8 hours. **b)** 3 hours. **c)** 2 hours. **d)** None of the previous answers is correct.

11. The solution f of the differential equation $y' = -3y + 7$ satisfying $f(0) = 1$ is the function defined on $\mathbb{R}$ by :

a) $f(x) = e^{-3x}$
b) $f(x) = -\frac{4}{3}e^{-3x} + \frac{7}{3}$
c) $f(x) = e^{-3x} + \frac{7}{3}$
d) None of the previous answers is correct.

12. Consider the function f defined on $\mathbb{R}$ by $f(x) = (ax+b)e^{-x} + 3$ where a and b are two constant real numbers.
The curve *(C)* of f is shown on the figure below.
Le point $A(0\,;1)$ belongs to the curve *(C)*. The line d tangent to the curve *(C)* at A passes through the point $B(3\,;4)$.
Then :

a) $a = 1$ and $b = -2$.
b) $a = -1$ and $b = -2$.
c) $a = 1$ and $b = 2$.
d) None of the previous statements is correct.



13. The limit, as n tends to $+\infty$, of the sequence (U_n) defined by $U_n = \frac{1+2^n}{3+3^n}$ equals :

a) 1 **b)** $\frac{2}{3}$ **c)** $+\infty$ **d)** None of the previous answers is correct.

14. Consider a function h , defined and continuous on $\mathbb{R}$, whose table of variations is given below.
Let H be the primitive of h, defined on $\mathbb{R}$, such that $H(0) = 0$.
H satisfies the following assertion:

a) H is positive on $]-\infty\,;0]$
b) H is increasing on $]-\infty\,;1]$
c) H is negative on $]-\infty\,;1]$
d) H is increasing on $\mathbb{R}$

x	$-\infty$		1		$+\infty$
Variations of h	$-\infty$	↗	0	↗	$+\infty$

15. A disease affects 20% of a country's population. When screening for this disease, we use a biological test that has the following characteristics:

- when a person is sick, the probability of having a positive test is 85%.
- when a person is not sick, the probability of having a negative test is 95%.

We select a person at random from this population. The probability for this person to have a positive test equals:

a) 85% **b)** 90% **c)** 21% **d)** None of the previous answers is correct.

16. The solution set of the following inequality $\sin(x) < \frac{1}{2}$ on the interval $[0\,;2\pi]$ is :

a) $[\frac{\pi}{6}\,;\,\frac{5\pi}{6}]$ **b)** $[0;\,\frac{\pi}{6}]$ **c)** $\left[0;\,\frac{\pi}{6}\right] \cup [\pi; 2\pi]$ **d)** None of the previous answers is correct.

17. We have a perfect cubic die numbered from 1 to 6, and an urn containing 5 indistinguishable balls of which 2 are white balls and 3 are black balls. We roll the die. If we obtain a number multiple of 3 then we draw 2 balls at random from the urn, successively with replacement. Otherwise, we randomly draw 2 balls from the urn simultaneously.

Consider then the following events:

M: "The roll of the die gives a number multiple of 3" , B: "The 2 balls drawn are white".

Hence:

a) $p(\mathrm{B}) = \frac{13}{100}$ **b)** $p(\mathrm{B}) = \frac{13}{50}$ **c)** $p(\mathrm{B}) = \frac{3}{25}$ **d)** None of the previous answers is correct.

18. Consider the following differential equation (E) : $y' - 2y = \cos(x) + 2\sin(x)$.

Let f be the unique solution of (E) satisfying $f(0) = 1$. The tangent line to the curve of f at the point of abscissa 0 is of equation :

a) $x = 3y + 1$ **b)** $y = 3x + 1$ **c)** $y = 1$ **d)** None of the previous answers is correct.

19. Let $z = 1 + e^{\frac{i\pi}{5}}$.

The modulus of z is equal to:

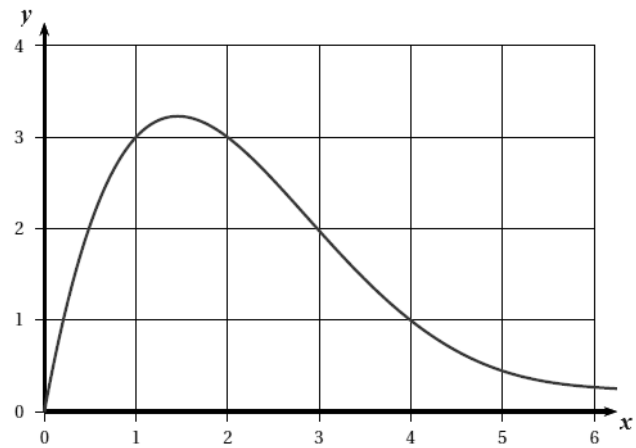
a) $2\cos\left(\frac{\pi}{5}\right)$ **b)** $\sqrt{1 + \left(\cos\left(\frac{\pi}{5}\right)\right)^2}$ **c)** $2\cos\left(\frac{\pi}{10}\right)$ **d)** None of the previous answers is correct.

20. The graph of a function f is shown on the figure below.

A bracketting of the integral $\int_1^5 f(x)dx$ is :

a) $0 \le I \le 4$ **b)** $1 \le I \le 5$

c) $5 \le I \le 10$ **d)** $10 \le I \le 15$



21. Consider in the set of complex numbers $\mathbb{C}$ the equation (E) : $\mathrm{z}^2 - 3(\sqrt{3} + \mathrm{i})\mathrm{z} + 4(1 + \mathrm{i}\sqrt{3}) = 0$ where the unknown is z. Let S be the solution set of (E) in $\mathbb{C}$.

a) $(\sqrt{3} + \mathrm{i})^2 = 2 - 2\mathrm{i}\sqrt{3}$. **b)** $\mathrm{S} = \{\,2(\sqrt{3} + \mathrm{i})\,;\ \sqrt{3} + \mathrm{i}\,\}$

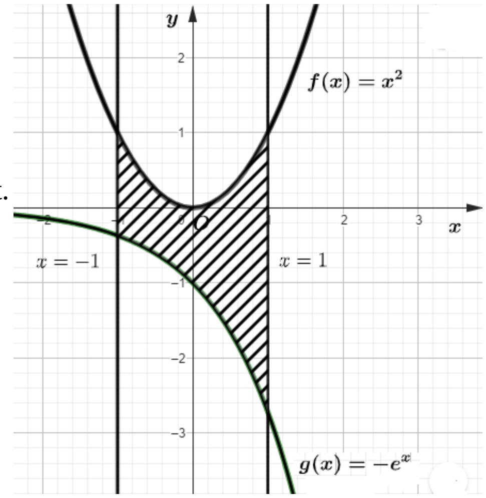
c) $\mathrm{S} = \{\,2(\sqrt{3} - \mathrm{i})\,;\ \sqrt{3} - \mathrm{i}\,\}$ **d)** None of the previous statements is correct.

22. A manufacturing line produces mechanical items. It is estimated that 4% of the items produced by this chain are defective. We randomly select n items produced by the production line. The number of items produced is large enough for this selection to be considered as a draw with replacement.

What is the smallest natural number n such that the probability of drawing at least one defective item is greater than or equal to 95%?

a) 50 **b)** 73 **c)** 74 **d)** None of the previous answers is correct.

23. The area of the hatched domain shown on the figure is:



a) $\frac{2}{3} - e + e^{-1}$ u.a. **b)** $\frac{2}{3} + e - e^{-1}$ u.a.

c) $-\frac{2}{3} + e - e^{-1}$ u.a. **d)** None of the previous answers is correct.

24. Consider the equation $e^{-2x} + 1 = 2e^{2x}$ of real unknown x.

a) This equation admits two solutions in ℝ. **b)** This equation admits only one solution in ℝ.
c) This equation admits no solution in ℝ.. **d)** None of the previous statements is correct.

25. $\frac{\sqrt{e^{-4} - 2e^{-2} + 1}}{(e^2)^3(e^{-3})^4} =$

a) $e^4 - e^6$ **b)** $e^6 - e^4$ **c)** $e^4 + e^6$ **d)** None of the previous answers is correct.

26. The profit of a company in thousands of dollars, made after the sale of a product, is given by the function defined by $B(x) = \frac{2(x-1)}{x^2 - 2x + 2}$ where the quantity produced x, in hundreds of units, is between 1 and 11.
Then the maximum profit achievable by the company is:

a) 1000 \$ **b)** 2000 \$ **c)** 6000 \$ **d)** 198 \$

27. Let A and B be two independent events of the same universe Ω such that $p(\bar{A}) = 0.6$ and $p(A \cup B) = 0.8$.
The probability of event B equals :

a) $\frac{2}{5}$ **b)** $\frac{2}{3}$ **c)** $\frac{1}{2}$ **d)** None of the previous answers is correct.

28. During a competition, the winner has the choice between two prizes:
- Prize A: he receives 2000 dollars per day for 15 days;
- Price B: he receives 1 dollar on the 1st day, 2 dollars on the 2nd day, 4 dollars on the 3rd day and so on, for 15 days the amount received doubles every day.

a) The value of prize A is higher than the value of prize B.
b) The value of prize B is higher than the value of prize A.
c) The value of prize B is twice as high as the value of prize A.
d) The value of prize A is twice as high as the value of prize B.

29. $\lim_{x \to 0^+} \frac{\sqrt{\ln(1+x^2)}}{x} =$

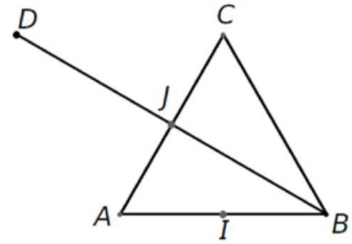
a) 0 **b)** $\frac{1}{2}$ **c)** 1 **d)** None of the previous answers is correct.

30. Consider an equilateral triangle ABC of side 2 such that $(\widehat{\overrightarrow{AB},\overrightarrow{AC}}) \equiv \frac{\pi}{3} \quad (mod\ 2\pi)$

Let I and J the midpoints of the segments $[AB]$ and $[AC]$ respectively.

D is the symmetric of the point B with respect to line (AC).

Let s be the direct similitude that transforms A into D, and I into J.



a) s is a similitude of ratio $\frac{1}{\sqrt{3}}$ and of angle $\frac{\pi}{6}$.

b) s is a similitude of ratio $\sqrt{3}$ and of angle $\frac{\pi}{6}$.

c) s is a similitude of ratio $\sqrt{3}$ and of angle $-\frac{\pi}{6}$.

d) None of the previous statements is correct.

<u>Exercice 2</u> : (8 points)

A municipality decided to install a skateboard module in a local park. The figure below provides a perspective view. The quadrilaterals OAD′D, DD′C′C and OAB′B are rectangles. The front plane (OBD) is provided with an orthonormal coordinate system (O, I, J). The unit is the meter.

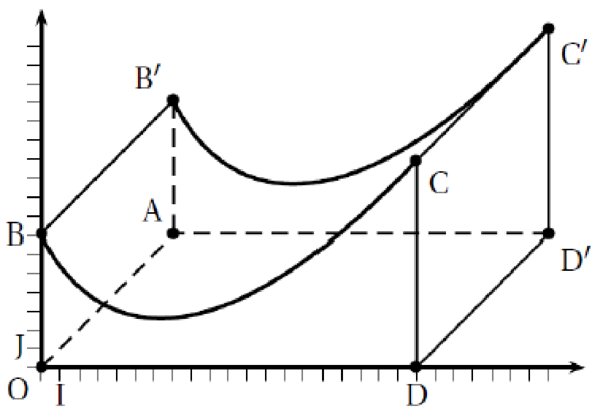
The width of the module is 10 meters, in other words DD′ = 10, and its length OD is 20 meters.

We would like to paint this module. **The goal of this problem is to then determine the area of the different surfaces to be painted.**

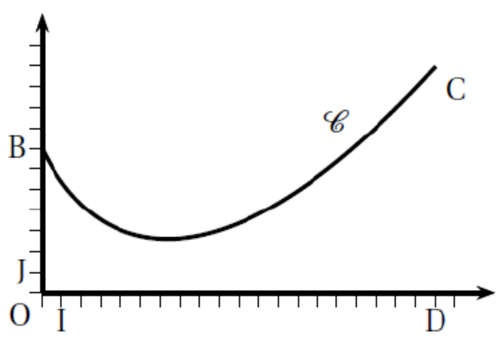
The profile of the skateboard module was modeled, from a photo, by the function f defined on the interval [0; 20] by:

$f(x) = (x+1)\ln(x+1) - 3x + 7$.

f' is the derivative function of the function f and $\mathscr{C}$ is the curve representing f in the orthonormal system (O, I, J).



Part A : *Profile study.*



1. Show that for all $x \in [0; 20]$, we have $f'(x) = \ln(x+1) - 2$.
2. Deduce the variations of f on [0 ; 20] and erect the table of variation of f.
3. Calculate the slope of the tangent line to the curve $\mathscr{C}$ at the point of abscissa 0.
   The absolute value of this slope is called *inclination* of the skateboard module at point B.
4. Study the convexity of the function f.
5. Are the following statements true ? Justify your answers.
   P_1 : The height difference between the highest point and the lowest point of the track is at least 8 meters.
   P_2 : The inclinaison of the track at B is almost twice the inclination at C.

6. Is there a point on the curve $\mathscr{C}$ where the tangent is parallel to line (BC) ? If so, find the coordinates of this point.
7. Show that the equation $f(x) = 7$ admits two solutions in the interval [0 ; 20], one of which is >15. This latter solution will be denoted α.
8. Consider the sequence (U_n) defined by $U_0 = 20$ and $U_{n+1} = h(U_n)$ for all integer $n \in \mathbb{N}$, where h is the function defined on $[\alpha; 20]$ by $h(x) = \frac{x-\ln(1+x)}{\ln(1+x)-2}$.

   a) Prove that the function h is increasing on the interval $[\alpha; 20]$.
   b) Using a proof by induction (recurrence), prove that for all integer $n \in \mathbb{N}$, we have : $\alpha \leq U_{n+1} \leq U_n \leq 20$.
   c) Deduce that the sequence (U_n) is convergent. What is its limit ?
   d) The following table gives the first terms of the sequence (U_n), rounded with 6 decimal places.

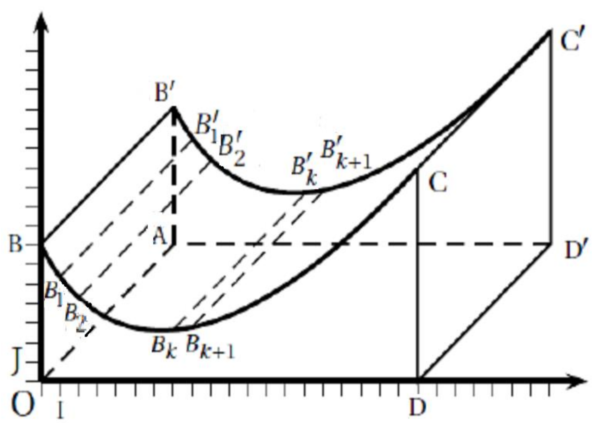
n	0	1	2	3	4	5
U_n	20.000000	16.232756	15.807457	15.801018	15.801016	15.801016

   Based on the values of this table, deduce an approximate value of α with 3 decimal places.

9. Show that the function g defined on [0 ; 20] by $g(x) = \frac{1}{2}(x+1)^2\ln(x+1) - \frac{1}{4}x^2 - \frac{1}{2}x$ is a primitive of the function $x \longmapsto (x+1)\ln(x+1)$ on the interval [0 ; 20].
10. Deduce the value of the integral $I = \int_0^{20} f(x)dx$. What is the mean height M of the profile $\mathscr{C}$?

Part B : ***Painting the module.***
The two questions of this part can be answered independently.

1. We want to cover the four side faces of this module with a layer of red paint. The paint used covers an area of 5 m^2 per liter. Then determine the minimum number of liters of paint needed (round to the nearest whole liter).
2. We now want to paint in black the rolling track, meaning the upper surface of the module.
   In order to determine an approximate value of the area of the part to be painted, we consider in the system (O, I, J) of the front plane, the points $B_k(k; f(k))$ for k varying from 0 to 20. So, B_0 = B.



We decide to approach the arc of the curve $\mathscr{C}$ joining B_k to B_{k+1} , by the segment $[B_k B_{k+1}]$. So the area of the surface to be painted can be approximated by the sum of the areas of the rectangles $B_kB_{k+1}B'_{k+1}B'_k$ (see figure).

   a) Show that for all integer k varying from 0 to 19, we have : $B_kB_{k+1} = \sqrt{1 + [\,f(k) - f(k+1)\,]^2}$.
   b) Numerical calculation has given the following sum : $S = \sum_{k=0}^{20} \sqrt{1+\left(f(k+1) - f(k)\right)^2} = 25.9835$

   Deduce then an approximate value of the area of the upper surface of the module.
   Knowing that the black paint used covers an area of 5 m^2 per liter, are 47 liters of paint enough to paint the rolling track of this module?

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MATHEMATICS

Answers sheet (Exercise 1)

TEST A

(Lebanese Program)

Date : *10/07/2024*

Answer to each question, by circling only one of the letters a, b, c or d.

1	a	b	c	d	**16**	a	b	c	d
2	a	b	c	d	**17**	a	b	c	d
3	a	b	c	d	**18**	a	b	c	d
4	a	b	c	d	**19**	a	b	c	d
5	a	b	c	d	**20**	a	b	c	d
6	a	b	c	d	**21**	a	b	c	d
7	a	b	c	d	**22**	a	b	c	d
8	a	b	c	d	**23**	a	b	c	d
9	a	b	c	d	**24**	a	b	c	d
10	a	b	c	d	**25**	a	b	c	d
11	a	b	c	d	**26**	a	b	c	d
12	a	b	c	d	**27**	a	b	c	d
13	a	b	c	d	**28**	a	b	c	d
14	a	b	c	d	**29**	a	b	c	d
15	a	b	c	d	**30**	a	b	c	d