LEBANESE UNIVERSITY
FACULTY OF ENGINEERING

الجامعة اللبنانية
كلية الهندسة

Entrance Exam 2004 -2005 **Mathematics** **Duration: 3 hours**
On 17/07/20004

The grades are over 25

I- (4.5 pts) The plane is referred to an orthonormal system $(O ; \vec{i}\ ,\ \vec{j}\ )$ Consider the points A $(m, 0)$ and B $(0, n)$ where m and n are two real numbers. Let P be the point such that $\overrightarrow{OA} = 2\overrightarrow{BP}$.

1- Determine the coordinates (x, y) of P in terms of m and n.

2- Suppose that m and n vary such that $AB = 2$.

a- Prove that the sets of points P is the ellipse (E) of equation $4x^2 + y^2 = 4$

b- Determine the focal axis, the vertices, the foci and the directress of (E). Construct (E).

3-Let (C) be the curve of equation $5x^2 + 6xy + 5y^2 = 8$

a-Prove that (C) is the transform of (E) by the rotation r of center O and angle $\frac{\pi}{4}$.

b-Deduce that (C) is a conic whose nature is to be determined.

c-Determine the focal axis and a focus of (C). Calculate the eccentricity and the area of (C).

II- (3.5 pts) Given n urns $U_1, U_2, \ldots, U_n$ where n is a natural number such that $n \geq 2$. The urn U_1 contains 2 black balls and one red ball and each of the other urns contains 1 black ball and 1 red ball.
We draw at random a ball from U_1 and we put in U_2 , then we draw a ball from U_2 and we put in U_3 then we draw a ball from U_3 and so on. Let E_k be the event "*the ball drawn from U_k is red*" and $\overline{E_k}$ the event opposite to E_k and denote by p_k the probability of E_k : $p_k = p\,(E_k)$.

1) Determine $p\,(E_1)$, $p\,(E_2/\,pE_1)$ and $p\,(E_2/\,p\,\overline{E_1})$ and prove that $p_2 = \frac{4}{9}$.

2) Prove that for all natural number k such that $1 \leq k \leq n$, $p_{k+1} = \frac{1}{3}p_k + \frac{1}{3}$

3) Consider the sequence (V_k) defined by $V_k = p_k - \frac{1}{2}$ *with* $k \geq 1$

a- Calculate V_1 and prove that (V_k) is a geometric sequence.

b- Calculate p_k in terms of k. Prove that the sequence (p_k) is convergent and calculate its limit.

III- (8 pts) Parts A, B and C of the problem are independent.

The complex plane is referred to an orthonormal system $(O ; \vec{u}\ ,\ \vec{v}\ )$ Consider the transformation T that, to each point M of affix z, associates the point M' of affix z' such that $z' = a z + b$ where a and b are two complex numbers such that $a \neq 0$, $b \neq 0$, and $a \neq b$.

A- Suppose in this part that $a = \dfrac{3}{4}$ and $b = \dfrac{1}{4}$

1) Determine the nature and the characteristic elements of T.
2) Determine the nature and the characteristic elements of T^{-1}

B- Suppose in this part that $a = 1 + i$ and $b = -i$

1) Determine the nature and the characteristic elements of T. Let w be the invariant point of T.
2) Consider the sequence of points M_n defined by M_0 (which is a point of the axis of abscissas with affix z_0) and $M_n = T(M_{n-1})$ and the sequence of their affixes z_n defined by:

   $z_0 = x_0 > 1$ and $z_n = (1+i)\, z_{n-1} - i$. Let $W_n = z_n - 1$.

   a- Prove that the sequence of general term W_n is a geometric sequence whose common ratio is to be determined.

   b- Calculate W_n in terms of x_0 and n.

   c- Calculate the modulus and an argument of W_n.

   d- Determine the values of the natural number n for which M_n of affix z_n is a point of the axis of abscissas.

   e- Determine x_0 so that M_4 is confounded with the origin O of the reference system.

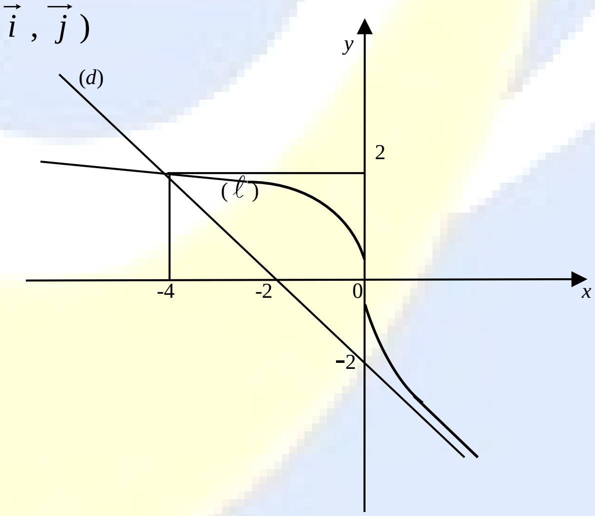
   f- Plot in this case the points M_0, M_1, M_2, M_3, and M_4.

C- Consider in this part the transformation T whose complex expression is $z' = a z + b$ and the transformation S whose complex form is $z' = b z + a$ such that $T \circ S = S \circ T$.

1) Prove that a and b are the roots of the equation $z^2 - z + m = 0$ where m is a number to be determined.
2) Prove that T, S and $S \circ T$ have the same invariant point which is a fixed point.

IV- (9 pts) The plane is referred to and orthonormal system $(O\,;\,\vec{i}\,,\,\vec{j}\,)$

A-Let h be a function defined on R whose representative curve (ℓ) is given in the adjacent figure such that (ℓ) is tangent at O to $y'Oy$ and admits (d) as an asymptote at $+\infty$ and $x'Ox$ as an asymptotic direction.



1) Prove that h admits on R an inverse function g such that $g(0) = 0$.
2) let (γ) be the representative curve of g .

   a- Determine the tangent to (γ) at point O and deduce $g'(0)$.

   b- Prove that (d) is an asymptote to (γ) and determine the point of intersection of (γ) and (d).

   c- Draw (γ) in a new reference system.

   d- Prove that $g(x)$ and x have opposite signs.

3) Suppose that g is defined on R by $g(x) = (ax + b)(1 + e^x) + c$, , where a , b and c are three real numbers.

a- Calculer $g'(x)$.

b- Using the values of $g(0)$, $g'(0)$ and $g(2)$ determined above, calculate a, b and c and verify that $g(x) = (2 - x)\, e^x - x - 2$

B- Consider the differential equation (E) : $(1 + e^x)\, y' - y = 0$ ..

1) Knowing that $\dfrac{1}{1+e^x} = \dfrac{e^{-x}}{1+e^{-x}}$, calculate $\displaystyle\int \frac{dx}{1+e^x}$

2) Solve the differential equation (E). Determine the particular solution of (E) whose representative curve passes through the point $I\,(0\,;\,2)$.

C- Let f be the function defined on R by $f(x) = \dfrac{4e^x}{1+e^x}$, and designates by (C) its representative curve.

1) Study the variations of f . Prove that f has an inverse function f^{-1} whose domain of definition is to be determined and calculate $f^{-1}(x)$.

2) Prove that the point $I\,(0\,;\,2)$ is a center of symmetry of (C) and determine an equation of the tangent (T) to (C) at point I .

3) Using the function g defined in part A , study the relative position of (C) and (T) .

4) Draw (C) and (T) .

D- Define on R , the function F by $F = g \circ f$.

1) Prove that F is decreasing.

2) Calculate $F(0)$ and the limit of F at $-\infty$.

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I- $A\,(m\,,0)$, $B\,(n\,,0)$, $P(x;\,y)$ and $\vec{OA} = 2\vec{BP}$

Then, $2((x_p - x_B) = x_A$ and $2(y_p - y_B) = y_A$ which gives $x = \frac{m}{2}$ and $y = n$

2) a- $AB^2 = 4$; then $m^2 + n^2 = 4$

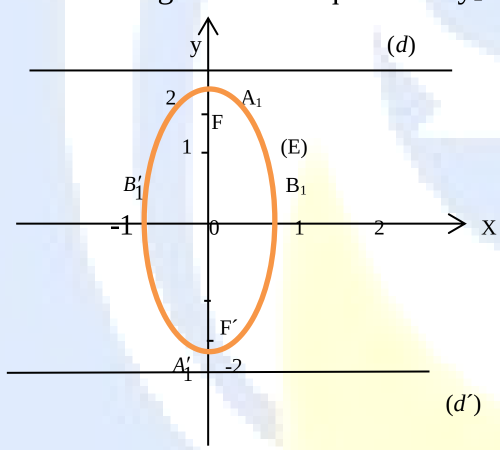
and consequently $(2x)^2 + (y)^2 = 4$. So, the set of the points P is the ellipse (E) of equation $4x^2 + y^2 = 4$.

b) The equation $4x^2 + y^2 = 4$ can be written as (E) : $x^2 + \frac{y^2}{4} = 1$

The focal axis is (y′ y), the principal vertices are A_1 (0; 2), A_1' (0; -2), the secondary vertices are B_1(1; 0) and B_1'(-1; 0)

$c^2 = a^2 - b^2 = 3$, the foci are F $(0, \sqrt{3})$ and F′ $(0, -\sqrt{3})$

The directress is the straight lines of equations: $y_1 = \frac{a^2}{c} = \frac{4\sqrt{3}}{3}$ and $y_2 = -\frac{a^2}{c} = -\frac{4\sqrt{3}}{3}$



3) a- the complex form of r is $z' = e^{i\frac{\pi}{4}}z$ which gives $z = e^{-i\frac{\pi}{4}}z'$;

If $M(x, y)$ is a point of (E) and $M'(x, y)$ its image by r then

$$x+iy = e^{-i\frac{\pi}{4}}(x'+iy') = \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)(x'+iy')$$

$$= \frac{\sqrt{2}}{2}(1-i)(x'+iy') = \frac{\sqrt{2}}{2}(x'+iy'-ix'+y') \text{ Therefore}$$

$x = \frac{\sqrt{2}}{2}(x'+y')$, $y = \frac{\sqrt{2}}{2}(y'-x')$ and as $x^2 + \frac{y^2}{4} = 1$

we have $\frac{1}{2}(x'+y')^2+\frac{1}{2}\frac{(y'-x')^2}{4}=1$

Let $4(x'+y')^2+(y'-x')^2=8$ and hence $5x'^2+6x'y'+5y'^2=8$

then the image of (E) by r is the curve (C) of equation: $5x^2+6xy+5y^2=8$

b) (E) is an ellipse and since rotation preserves geometric figures then (C) is an ellipse.

c) The focal axis (Δ) of (C) is the image of the focal axis of (E) by r. But the focal axis of (E) is the axis $y'y$ of equation $x = 0$

$x = 0$ gives $\frac{\sqrt{2}}{2}(x'+y') = 0$ then the straight line (Δ) of equation $y=-x$ is the focal axis of (C)

F (0, $\sqrt{3}$) is a focus of (E), this point F is transformed onto a point F_1 by the rotation r

$z_{F_1}=e^{i\frac{\pi}{4}}z_F \quad =\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)(i\sqrt{3})=\frac{i\sqrt{6}}{2}-\frac{\sqrt{6}}{2}$ therefore F_1 $(-\frac{\sqrt{6}}{2};\frac{\sqrt{6}}{2})$ is a focus of (C)

The eccentricity of (C) is equal to that of (E), therefore, $e = \frac{\sqrt{3}}{2}$.

Area of (C) = area of (E) = $\pi\, ab = 2\pi$ *square units*

II- 1) $P(E_1) = \frac{1}{3}$ E_1 has occurred , then urn U_2 contains two red balls and one black ball, then

$P\left(\frac{E_2}{E_1}\right)=\frac{2}{3}$.

$\overline{E_1}$ has occurred, then urn U_2 contains one red balls and two black balls, then $P\left(\frac{E_2}{\overline{E_1}}\right)=\frac{1}{3}$

$P(E_2) = P_2 = P(E_2 \cap E_1) + P(E_2 \cap \overline{E_1})$

$=P(E_1) \times P\left(\frac{E_2}{E_1}\right) + P(\overline{E_1}) \times P\left(\frac{E_2}{\overline{E_1}}\right) = \frac{1}{3}\times\frac{2}{3}+\frac{1}{3}\times\left(1-\frac{1}{3}\right)=\frac{4}{9}$

2) $P_{k+1}=P(E_{k+1})=P(E_k\cap E_{k+1})+P(\overline{E}_k\cap E_{k+1})$

$= p\left(\frac{E_{k+1}}{E_k}\right)\times p(E_k)+p\left(\frac{E_{k+1}}{\overline{E_k}}\right)\times p(\overline{E_k})$ or

$p\left(\frac{E_{k+1}}{E_k}\right)=\frac{2}{3}$ and $p\left(\frac{E_{k+1}}{\overline{E_k}}\right)=\frac{1}{3}$

$$P_{k+1} = P(E_{k+1}) \; = \frac{2}{3}\times P_k + \frac{1}{3}\times(1-P_k) = \frac{1}{3}P_k + \frac{1}{3}$$

3) a- $V_1 = P_1 - \frac{1}{2} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$

$$V_{k+1} = P_{k+1} - \frac{1}{2} = \frac{1}{3}P_k + \frac{1}{3} - \frac{1}{2} = \frac{1}{3}P_k - \frac{1}{6} = \frac{1}{3}(P_k - \frac{1}{2}) = \frac{1}{3}V_k$$

Then V_k is a geometric sequence of common ratio 1 /3

b- $V_k = V_1 \times q^{k-1} = -\frac{1}{6}\left(\frac{1}{3}\right)^{k-1}$ *where* $P_k = V_k + \frac{1}{2} = -\frac{1}{6}\left(\frac{1}{3}\right)^{k-1} + \frac{1}{2}$

The sequence (V_k) is bounded above by $\frac{1}{2}$ since $-\frac{1}{6}\left(\frac{1}{3}\right)^{k-1} + \frac{1}{2} < \frac{1}{2}$

Also, the sequence (V_k) is increasing since

$$P_{k+1} - P_k = \frac{1}{6}\left(\frac{1}{3}\right)^{k-1} - \frac{1}{6}\left(\frac{1}{3}\right)^{k} = \frac{1}{6}\left(\frac{1}{3}\right)^{k-1}\left(1-\frac{1}{3}\right) = \frac{2}{3}\times\frac{1}{6}\left(\frac{1}{3}\right)^{k-1} > 0$$

Then $P_{k+1} > P_k$.

$$\lim_{k\to+\infty} P_k = \frac{1}{2} \text{ since } \lim_{k\to+\infty}\left(\frac{1}{3}\right)^{k-1} = 0$$

III- A- 1) T is of the form $z' = (\frac{3}{4}z + \frac{1}{4})$ with $a = \frac{3}{4}$, which is a real number, hence T is a dilation of ratio $\frac{3}{4}$ and center the point w of affix $z_w = \frac{b}{1-a} = 1$.

2) T^{-1} is a dilation of ratio $\frac{4}{3}$ and the same center as T .

B- 1) T is of the form $z' = (1+i)z - i$ *with* $a = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$

T is a similitude of ratio $\sqrt{2}$, angle $\frac{\pi}{4}$,and of center the point w of affix

$$z_w = \frac{b}{1-a} = \frac{-i}{-i} = 1$$

2) a- $W_{n+1} = z_{n+1} - 1 = (1+i)z_n - i - 1 = (1+i)(z_n - 1) = (1+i)W_n$

The sequence of general term W_n is a geometric sequence of ratio $q = 1+i$ and a first term

$W_0 = z_0 - 1 = x_0 - 1$

b- $W_n = W_0 \times q^n = (x_0 - 1)(1+i)^n$

c- $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$, *where* $|W_n| = |x_0 - 1| \times |1+i|^n$ *or* $x_0 - 1 > 0$ *where* $|W_n| = (x_0 - 1) \times (\sqrt{2})^n$.

$\arg W_n = \arg\left((x_0 - 1) \times (1+i)^n\right) = \arg(x_0 - 1) + \arg(1+i)^n$

$\arg W_n = 0 + n\frac{\pi}{4}(2\pi) = n\frac{\pi}{4}(2\pi)$

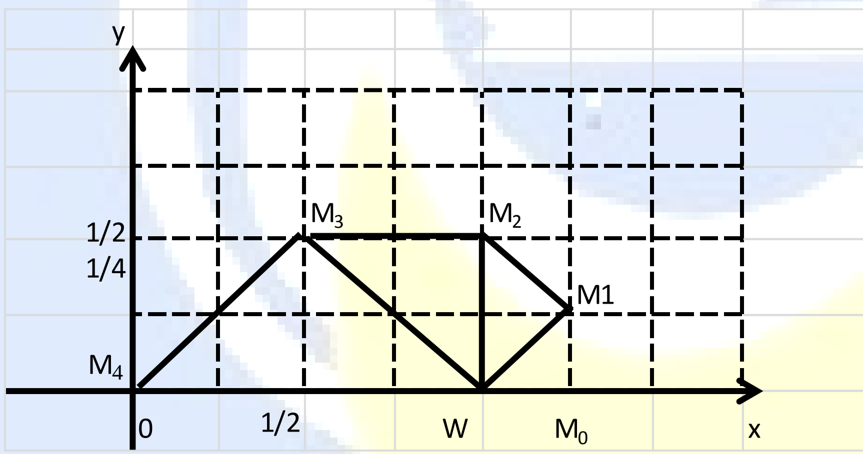
d- If M_n is a point of the axis of abscissas axis then z_n is real, so $W_n = z_n - 1$ is real

such that $n\frac{\pi}{4} = k\pi$ and $n = 4k$ with $k \in IN$ since k is a natural number.

e- if M_4 is confounded with O then $z_4 = 0$, where $W_4 = -1$, then

$(x_0 - 1) \times (1+i)^n = -1$, *and* $x_0 - 1 = \frac{-1}{(1+i)^4} = \frac{1}{4}$ *then* $x_0 = \frac{5}{4}$

f-



C- 1) $M(z) \xrightarrow{s} M'(z' = bz + a) \xrightarrow{T} M''(z'' = az + b)$

$M(z) \xrightarrow{T \circ s} M''(z'' = abz + a^2 + b)$

$M(z) \xrightarrow{T} M_1(z_1 = az + b) \xrightarrow{s} M_2(z_2 = bz_1 + a)$

$M(z) \xrightarrow{s \circ T} M_2(z_2 = abz + b^2 + a)$

$T \circ S = S \circ T$ gives $T \circ S(M) = S \circ T(M)$ then

$abz + a^2 + b = abz + b^2 + a$, consequently $a^2 - b^2 = a - b$

$(a-b)(a+b) - (a-b) = 0$ or $(a-b)(a+b-1) = 0$

$a - b = 0$ or $a + b - 1 = 0$ and hence $a \neq b$ we get $a + b = 1$

Hence a and b are the roots of the equation $z^2 - z + m = 0$ where $m = ab$

with $m \neq 0$ since $a \neq 0$ and $b \neq 0$

2) $z_{w_T} = \dfrac{b}{1-a} = \dfrac{1-a}{1-a} = 1,\quad z_{w_S} = \dfrac{a}{1-b} = \dfrac{1-b}{1-b} = 1$

$z_{w_{S \circ T}} = \dfrac{b^2+a}{1-ab} = \dfrac{b^2+1-b}{1-b(1-b)} = 1$, then T, S and $S \circ T$ have the same double point.

IV) A)

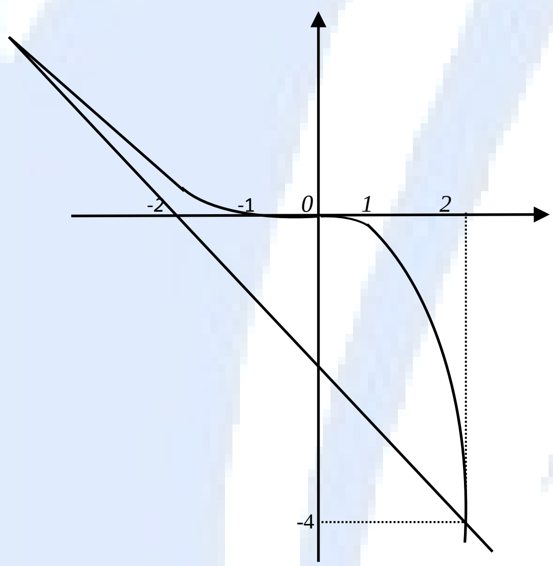
1) h is continuous and strictly decreasing over IR then it admits an inverse function g over IR
Since (ℓ) passes through O then g passes through O hence g(0) = 0

2) Let (γ) be the representative curve of g

a- The axis y´y is tangent to (ℓ) at O, then the curve (γ) is tangent to the axis $x'x$ at O, consequently $g'(0) = 0$

b- The straight line (d) is an asymptote to (ℓ), (γ) has as an oblique asymptote the straight line (d') symmetric of (d) with respect to the straight line of equation y = x.
But, (d) passes through the point (0 ; -2) , (d') passes through the point (-2 ; 0) similarly, (d) passes through the point (-2 ; 0) so (d') passes through the point (0 ; -2) , (d') is (d) itself consequently (d) is an asymptote of (γ)

c-



d- From the representative curve of g, we note that a part of (γ)

is situated in the first quadrant and a part in the fourth quadrant and in both cases $g(x)$ and x have opposite signs except for O where g(0) = 0

3) a- $g'(x) = a(1+e^x) + e^x(ax+b)$

b- g(0) = 0 gives $2b + c = 0$

$g'(0) = 0$ gives $2a + b = 0$

$h(-4) = 2$, and $g(2) = -4$ which gives $(2a + b)(1 + e^2) + c = -4$

But $2a+b = 0$ so $c = -4$; $2b + c = 0$

Then b = 2, 2a+b = 0 , a = -1 so

$g(x) = (-x+2)(1+e^x) - 4$, consequently $g(x) = (2-x)e^x - x - 2$

B) 1) $\int \frac{dx}{1+e^x} = \int \frac{e^{-x}}{1+e^{-x}}dx = -\ln(1+e^{-x}) + k$

2) (E) : $(1+e^x)y' - y = 0$ is equivalent to

$\frac{y'}{y} = \frac{1}{1+e^x}$ $where \int \frac{y'}{y}dx = \int \frac{1}{1+e^x}dx, then$ $\ln|y| = -\ln(1+e^{-x}) + k$

Let $\ln|y| + \ln(1+e^{-x}) = k \; or \ln\left|y(1+e^{-x})\right| = k \; then \; y(1+e^{-x}) = \pm e^{k}$

A result $y = \pm\dfrac{e^{k}}{1+e^{-x}}$ .Consequently, $y = \dfrac{C}{1+e^{-x}}$

At the point I (0 ; 2) we have $2 = \dfrac{C}{1+1}$, which gives C = 4 and consequently $y = \dfrac{4}{1+e^{-x}} = \dfrac{4e^{x}}{1+e^{x}}$

C) 1) $f'(x) = \dfrac{4e^{x}}{(1+e^{x})^{2}} > 0$, $\lim\limits_{x \to -\infty} f(x) = 0$, $\lim\limits_{x \to +\infty} f(x) = 4$

x	$-\infty$ $\qquad\qquad\qquad\qquad$ $+\infty$
$f'(x)$	+
$f(x)$	0 $\nearrow$ 4

f is continuous and strictly increasing over *IR*, the nit admits an inverse functio
f^{-1} *whose domain is*]0, 4[

$y = \dfrac{4e^{x}}{1+e^{x}}$ gives $y + ye^{x} = 4e^{x}$ then e^{x}(4-y) = y , which gives $e^{x} = \dfrac{y}{4-y}$

$x = \ln\dfrac{y}{4-y}$ then $f^{-1}(x) = \ln\dfrac{x}{4-x}$

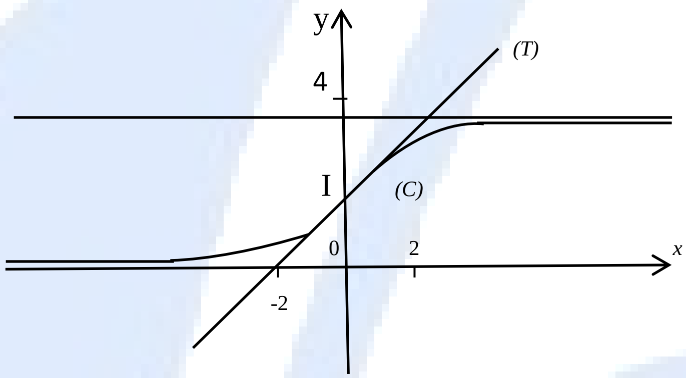
2) $f(-x) + f(x) = \dfrac{4e^{-x}}{1+e^{-x}} + \dfrac{4e^{x}}{1+e^{x}} = \dfrac{4e^{x}+4}{1+e^{x}} = \dfrac{4(e^{x}+1)}{1+e^{x}} = 4$ then the point *I* (0, 2) is a center of

symmetry for (*C*)

An equation of the tangent (*T*) to (*C*) at the point *I* is
y= f(0)+ f ´(0)(x-0) =2+1(x-0) let *y* = *x* + 2

3) $f(x) - y = \dfrac{4e^{x}}{1+e^{x}} - x - 2 = \dfrac{4e^{x} - x - xe^{x} - 2 - 2e^{x}}{1+e^{x}} = \dfrac{g(x)}{1+e^{x}}$ But, *x* and g(*x*) have opposite signs, where
if *x* > 0 , g(*x*) < 0 , (*C*) is below (*T*)
if *x* < 0 , g(*x*) > 0 , (*C*) is above (*T*)

4)



D- 1) $F(x) = g(f(x))$, $F'(x) = g'(f(x))$ x $f'(x)$, but g is decreasing so $g'(f(x)) < 0$ and f is strictly increasing $f'(x) > 0$ then $F'(x) < 0$ and F is decreasing.

2) $F(0) = g(f(0)) = g(2) = -4$

$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} g(f(x)) = g(0) = 0$$