Entrance Exam 2003 -2004 **Mathematics** **Duration: 3 hours**
July 2003

Remarks: - The use of the non-programmable calculator is allowed.
- The distribution of grades is over 25

I- (4 points) We admit that, for any natural number α $\lim\limits_{n\to\infty} x^{\alpha}e^{-x} = 0$

Let $U_p(n) = \int_0^n x^p e^{-x}dx$ where n and p are two natural numbers.

1) Calculate $U_0(n)$ and show that $U_1(n) = 1-(1+n)e^{-n}$

2) Using integration by parts, prove that $U_2(n) = 2U_1(n) - n^2e^{-n}$ .Calculate $\lim\limits_{n\to\infty} U_1(n)$ and deduce $\lim\limits_{n\to\infty} U_2(n)$

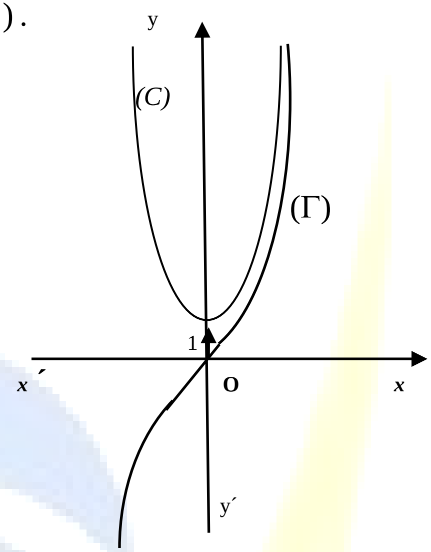
3) Using integration by parts, find a relation between $U_p(n)$ and $U_{p-1}(n)$ .Deduce that $\lim\limits_{n\to\infty} U_p(n) = p!$

II- (3 points) We are given 3 urns U_1, U_2 and U_3 such that: U_1 contains <u>one</u> red ball and 4 white balls; U_2 Contains 4 red balls and 4 white balls; and U_3 contains 7 red balls and 3 white balls.

Designate by: P_1 the probability that urn U_1 is chosen;
P_2 the probability that urn U_2 is chosen;
P_3 the probability that urn U_3 is chosen;

1) Knowing that P_1, P_2 and P_3 are respectively proportional to 1 , 2 and 3, prove that $P_1 = \frac{1}{6}$ and calculate P_2 and P_3.

2) An urn is chosen and a ball is selected at random from this urn.
 a) Calculate the probability to select a red ball knowing that it comes from U_1
 b) Calculate the probability of the event: “ ***the selected ball is red and it comes from*** U_1 “.
 c) Calculate the probability of the event: “ ***the selected ball is red***”.
 d) Knowing that the selected ball is red, what is the probability that it comes from U_1 ?



III- (8 points) the plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.
The adjacent 2 curves (C) and (Γ) represent respectively
The variations of 2 functions f and g , defined on *IR* , *such that*
f *is* the derivative of g and g is the derivative of f.

1) *Let* $h(x) = \ln[f(x) - g(x)]$.
   a) Prove that $h'(x)$ is constant.
   b) Deduce that $f(x) - g(x) = e^{-x}$
2) Designate by U_n the area of the domain bounded by (C), (Γ) and the 2 straight lines of equations $x = n - 1$ and $x = n$ where $n \in IN^{\bullet}$
   a) Prove that $U_n = (e-1)e^{-n}$
   b) Prove that U_n is the general term of a geometric sequence whose first term and common ratio are to be determined.
   c) Calculate in terms of n , the sum $S_n = U_1 + U_2 + \ldots\ldots + U_n$ and determine its limit as n tends to $+\infty$
   d) Determine the values of n such that $S_n > 0.99$. Let P be the least of these values; give a framing of S_p of amplitude 10^{-3} .
3) a) Prove that f and g are 2 solutions of the same differential equation (E) of second order which is to be determined .
   b) Solve (E)and deduce the expression of $f(x)$ and that of $g(x)$.
4) By only using the relation $f(x) - g(x) = e^{-x}$ and admitting that f is even and g is odd, prove that $f(-x) + g(-x) = e^{-x}$ and find again the expressions of $f(x)$ and $g(x)$

IV- (10 points) The parts *A* and *B* of the problem are independent.

In the complex plane referred to an orthonormal system $(O;\vec{u},\vec{v})$, consider the transformation T that associates to each point M of affix z the point M' of affix z' such that $z' = az + b$ where a and b are 2 complex numbers such that $a \neq 0$ and $a \neq 1$.

A- **Suppose in this part that** $b \neq 0$.
Consider the sequence of points M_n defined by $M_0 = O$ (O being the origin of the system) and $M_n = T(M_{n-1})$, and the sequence of their respective affixes z_n defined by $z_0 = 0$ and $z_n = az_{n-1} + b$

1) a) Prove by mathematical induction that , for all $n \geq 1$, $z_n = b\dfrac{1-a^n}{1-a}$
   b) Prove that if $|a| < 1$, z_n has a limit ℓ to be determined.
   c) What does the point L of affix ℓ represent for the transformation T ?

2) Suppose that $a = \cos 2\alpha + i \sin 2\alpha$ and $b = 2\sin\alpha$ where α is a number which is not a multiple of π

a) Give the nature of the corresponding transformation T and determine its characteristic elements in terms of α .

b) Deduce that the points M_n of affixes z_n belong to a circle passing through O whose radius and the coordinates of its center are to be determined.

c) Construct a figure in case where $\alpha = \frac{\pi}{3}$ and plot the points M_0, M_1, M_2 and M_3 .

B- **Suppose in this part that** $\alpha = 1 + i$ and $b = 0$. The transformation T will have the complex expression $z' = (1+i)\, z$.

1) What is the nature of T ? Determine its characteristic elements.

2) Consider the hyperbola (H) of equation $\frac{x^2}{4} - \frac{y^2}{5} = 1$

a) Determine the center of (H), its vertices and the equations of its asymptotes. Draw (H).

b) Determine the eccentricity of (H), one of its foci and the corresponding directrix .

3) Designate by (H') the transform of (H) by T .

a) Prove that the equation of (H') is $x^2 + y^2 + 18xy = 80$

b) Consider the point F_1 (3, 3) and the straight line (Δ) of equation $3x + 3y - 8 = 0$
Prove that the set of points N such that $4NF_1^2 = 9NK^2$ where NK is the distance from N to (Δ) , is the curve (H').

c) Deduce that (H') is a conic whose nature, eccentricity, a focus and the corresponding directrix are to be determined.

Entrance exam 2003-2004 **Solution of Mathematic** **Duration: 3 hours**

I- 1) $U_0(n)=\int_0^n e^{-x}dx=-e^{-x}\Big|_0^n=-e^{-n}+1, U_1(n)=\int_0^n xe^{-x}dx$

Letting $u=x$ and $v'=e^{-x}$, we get:
$u'=1$ and $v=-e^{-x}$, which gives

$$U_1(n)=\int_0^n xe^{-x}dx=-xe^{-x}\Big|_0^n+\int_0^n e^{-x}dx=-ne^{-n}-e^{-n}+1=1-(1+n)e^{-n}$$

2) $U_2(n)=\int_0^n x^2e^{-x}dx$

Letting $u=x^2$ and $v'=e^{-x}$, we get:
$u'=2x$ and $v=-e^{-x}$, which gives

$$U_2(n)=\int_0^n x^2e^{-x}dx=-x^2e^{-x}\Big|_0^n+\int_0^n 2xe^{-x}dx=-n^2e^{-n}+2\mathrm{U}_1(n)$$

$$\lim_{n\to+\infty}U_1(n)=\lim_{n\to+\infty}[(1-e^{-n})+(-ne^{-n})]=1-0-0=1$$

$$\lim_{n\to+\infty}U_2(n)=\lim_{n\to+\infty}2U_1(n)-n^2e^{-n}=2-0=2$$

3) $U_p(n)=\int_0^n x^pe^{-x}dx$

Letting $u=x^p$ and $v'=e^{-x}$, we get
$u'=px^{p-1}$ et $v=-e^{-x}$, which gives

$$U_p(n)=-x^pe^{-x}\Big|_0^n+p\int_0^n x^{p-1}e^{-x}dx=-n^pe^{-n}+p\mathrm{U}_{p\text{-}1}(n)$$

$$\lim_{n\to+\infty}U_p(n)=\lim_{n\to+\infty}[-n^pe^{-n}+pU_{p-1}(n)]=$$

$$-\lim_{n\to+\infty}n^pe^{-n}+p\lim_{n\to+\infty}U_{p-1}(n)=0+p\times\lim_{n\to+\infty}U_{p-1}(n)=p\times\lim_{n\to+\infty}U_{p-1}(n)$$

With the same reasoning, we get

$$\lim_{n\to+\infty}U_{p-1}(n)=(p-1)\lim_{n\to+\infty}U_{p-2}(n)$$

then $\lim_{n\to+\infty}U_p(n)=p\times(p-1)\times\lim_{n\to+\infty}U_{p-2}(n)$

As a result, we get :

$$\lim_{n\to+\infty}U_p(n)=p\times(p-1)\times\ldots\ldots\times2\times\lim_{n\to+\infty}U_1(n)$$
$$=p\times(p-1)\times\ldots\ldots\times2\times1=p!$$

II-1) p_1, p_2 and p_3 are proportional at 1, 2 and 3

therefore : $\frac{p_1}{1}=\frac{p_2}{2}=\frac{p_3}{3}=k$ and as $p_1+p_2+p_3=1$, we have :

$k+2k+3k=1$, and since $k=\frac{1}{6}$, and consequently :

$p_1=\frac{1}{6}, p_2=\frac{2}{6}=\frac{1}{3} and\ p_3=\frac{3}{6}=\frac{1}{2}$

2) a- $p(R/U_1)=\frac{1}{5}$; since U_1 contains 5 balls of which one only is red.

b- $p(R\cap U_1)=p(U_1)\times p(R/U_1)=\frac{1}{6}\times\frac{1}{5}=\frac{1}{30}$

c- $p(R)=p(R\cap\Omega)=p(R\cap(U_1\cup U_2\cup U_3))$

$=p((R\cap U_1)\cup(R\cap U_2)\cup(R\cap U_3))$

$=p(R\cap U_1)+p(R\cap U_2)+p(R\cap U_3)$

Or: $p(R\cap U_2)=p(U_2)\times p(R/U_2)=\frac{1}{3}\times\frac{4}{8}=\frac{1}{6}$

$p(R\cap U_3)=p(U_3)\times p(R/U_3)=\frac{1}{2}\times\frac{7}{10}$, where

$p(R_1)=p(R\cap U_1)+p(R\cap U_2)+p(R\cap U_3)=\frac{1}{30}+\frac{1}{3}\times\frac{4}{8}+\frac{1}{2}\times\frac{7}{10}=\frac{11}{20}=0{,}55$

d- : p (U_1/R)= $\frac{p(U_1\cap R)}{p(R)}=\frac{1}{30}\times\frac{20}{11}=\frac{20}{30\times 11}=\frac{2}{33}$

III-1) a- $h'(x)=\frac{[f(x)-g(x)]}{f(x)-g(x)}=\frac{f'(x)-g'(x)}{f(x)-g(x)}=\frac{g(x)-f(x)}{f(x)-g(x)}=-1$

Then $h'(x)$ is a constant, so $h(x)=-x+k$

b- Graphically, we notice that (C) passes through the point (0 ; 1) and (Γ) passes through the origin (0 ; 0), then $f(0)=1$ and $g(0)=0$, which gives : $h(0)=\ln[f(0)-g(0)]=\ln 1=0$ and on the other hand $h(0)=k$, therefore $k=0$ and consequently $h(x)=-x$.

Hence : $f(x)-g(x)=e^{-x}$

2) a- $U_n = \int_{n-1}^{n} [f(x) - g(x)dx] = \int_{n-1}^{n} e^{-x} dx = [-e^{-x}]_{n-1}^{n}$

$= -[e^{-n} - e^{-(n-1)}] = e^{-(n-1)} - e^{-n} = e^{-n+1} - e^{-n} = e^{-n}(e-1)$

b- $U_{n+1} = (e-1)e^{-(n+1)} = (e-1)e^{-n} \times e^{-1} = \frac{e-1}{e} e^{-n} = \frac{U_n}{e}$

Then, (U_n) is a geometric sequence of first term

$U_1 = 1 - e^{-1} = 1 - \frac{1}{e}$ and of common ratio $q = \frac{1}{e}$

c- $S_n = U_1 + U_2 + ... + U_n$ is the sum of n consecutive terms of geometric sequence of first term

$U_1 = (1 - \frac{1}{e})$ and of common ratio $q = \frac{1}{e}$

$$S_n = U_1 \frac{1 - q^n}{1 - q} = (1 - \frac{1}{e}) \frac{1 - (\frac{1}{e})^n}{(1 - \frac{1}{e})} = 1 - e^{-n}$$

$\lim_{n \to \infty} S_n = 1$

d- $S_n > 0.99$ gives $1 - e^{-n} > 0.99$, Let $e^{-n} < 0.01$, where:

then $-n < \ln(0,01)$, so $-n < \ln(\frac{1}{100})$, which gives $-n < -\ln 100$ and consequently $n > \ln(100)$

or $n > 4,605$.

That is $n \geq 5$, since, n is a natural number. The smallest of these values is then $p = 5$.

$S_5 = 1 - e^{-5} = 0,9932620$ A bounding of S_5 to the nearest 10^{-3} is then $0,993 < S_5 < 0,994$.

3) a- Since $f'(x) = g(x)$ and $g'(x) = f(x)$; we get $f''(x) = g'(x) = f(x)$ and consequently

$f''(x) - f(x) = 0$ or $y'' - y = 0$. Similarly,

$g''(x) = f'(x) = g(x)$; Which gives $y'' - y = 0$

Then, f and g are the solutions of the differential equation $y'' - y = 0$

b- The characteristic equation associated with the differential equation is $r^2 - 1 = 0$, that has as solutions
$r_1 = 1$ and $r_2 = -1$, then the general solution of (E) is $y = C_1 e^x + C_2 e^{-x}$.
But, $f(0) = 1$; gives $C_1 + C_2 = 1$

$y' = C_1 e^x - C_2 e^{-x}$ and since g(0) = $f'(0)=0$; $C_1 - C_2 = 0$ then $C_1 = C_2 = \frac{1}{2}$

$f(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ and since $g(x) = f'(x)$ we get $g(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

4) f is even , then $f(-x) = f(x)$ and g is odd, then $g(-x) = -g(x)$
$f(-x) + g(-x) = f(x) - g(x) = e^{-x}$ This relation gives $f(x) + g(x) = e^x$

We get then the two relations: $f(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ and $g(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

IV-A 1) a- $b \neq 0$, M_0 = O and $M_n = T(M_{n-1})$ for n = 1, we get

$z_1 = az_0 + b = a \times 0 + b = b = b \times \frac{1-a^1}{1-a}$, Then the relation is verified for $n = 1$

Suppose that the relation of order n is true. *We* have to prove that it remains true for the order n+1.

$$z_{n+1} = az_n + b = a[b\frac{1-a^n}{1-a}] + b = \frac{ab(1-a^n) + b(1-a)}{1-a} = \frac{b - ba^{n+1}}{1-a} = b\frac{1-a^{n+1}}{1-a}$$

The relation is true for all $n \geq 1$

b- If $|a| < 1$ *then* $\lim\limits_{n\to\infty} a_n = 0$ consequently $\lim\limits_{n\to\infty} z_n = \frac{b}{1-a}$ hence $\ell = \frac{b}{1-a}$

c- The point L (ℓ) is invariant under T.

2) a- The complex form of T is $z' = az + b$ then T is a direct plane similitude.
But $a = \cos 2\alpha + i\sin 2\alpha = e^{i2\alpha}$, $|a| = 1$ *and* $\arg(a) = 2\alpha$, the ratio of T is 1 and its angle is 2α. The center of T is an invariant point L, of affix

$$l = \frac{b}{1-a} = \frac{2\sin\alpha}{1-\cos 2\alpha - i\sin 2\alpha} = \frac{2\sin\alpha}{2\sin^2\alpha - 2i\sin\alpha\cos\alpha} = \frac{1}{\sin\alpha - i\cos\alpha} = \sin\alpha + i\cos\alpha$$

Hence, T is a rotation of center point L ($\sin\alpha;\cos\alpha$) and angle 2α.

b- T is a rotation of center L and of angle 2α

$M_0 \xrightarrow{T} M_1 \xrightarrow{T} M_2 .... M_{n-1} \xrightarrow{T} M_n$

$LM_0 = LM_1 = LM_2 = ... = LM_n$

But $LM_0 = LO$ because $M_0 = O$ consequently $OL = |\ell| = \sin^2\alpha + \cos^2\alpha = 1$

$LO = LM_1 = LM_2 = ... = LM_n = 1$

So, the points $M_n(z_n)$ belong to the same circle of centre L and radius 1. This circle passes through O since $LO = 1$.

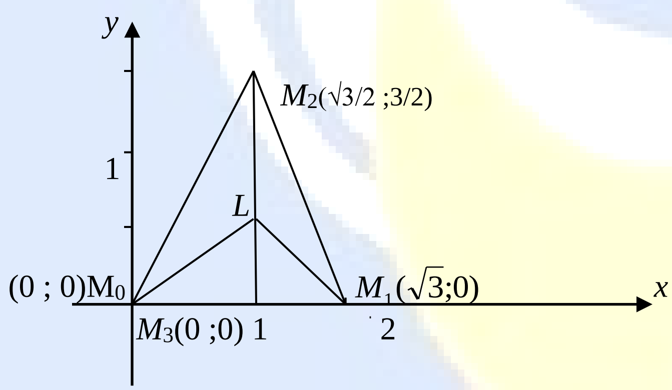
c- $\alpha = \dfrac{\pi}{3}$ gives: $a = \cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$

$b = 2\sin\left(\dfrac{\pi}{3}\right) = \sqrt{3}$ $where: z_0 = 0,\ z_1 = az_0 + b = \sqrt{3}$

$$z_2 = az_1 + b = \sqrt{3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \sqrt{3} = \frac{\sqrt{3}}{2} + \frac{3}{2}i$$

$$z_3 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2} + \frac{3}{2}i\right) + \sqrt{3} = 0$$

$M_0 = O,\ M_1(\sqrt{3};0)$, $M_2\left(\dfrac{\sqrt{3}}{2};\dfrac{3}{2}\right),\ M_3 = O$

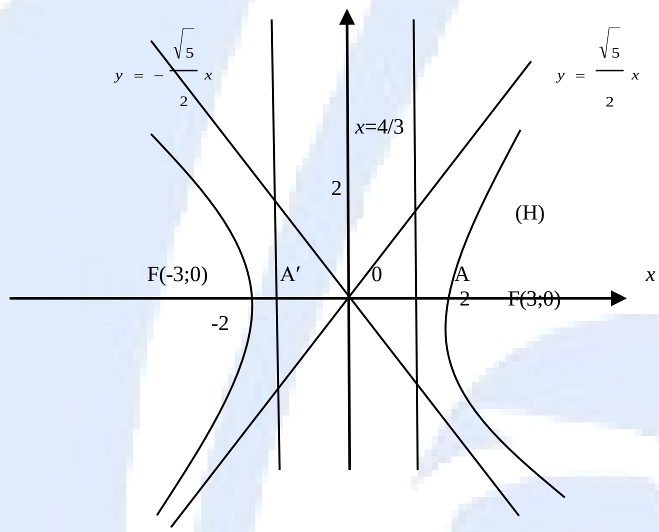


B. 1) $a = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$, then T is a direct plane similitude of center O, $b = 0$, of ratio $\sqrt{2}$ and angle $\dfrac{\pi}{4}$

2) The center of (H) is O (0; 0)

For $y = 0$;we get $\dfrac{x^2}{4} = 1,\ x = 2\ \ where\ \ x = -2$, then the vertices of ($H$) are A(2; 0) and A'(-2; 0)

The asymptotes of (H) are $y = \frac{\sqrt{5}}{2}x$ *and* $y = -\frac{\sqrt{5}}{2}x$



b- $c^2 = a^2 + b^2 = 4 + 5 = 9$; $c = 3$ where $e = \frac{c}{a} = \frac{3}{2}$

The foci of (H) are : $F(c;\ 0)$ and $F'(-c;\ 0)$ So they are F (3; 0) and F'(-3; 0)

The directrices are the lines of equations: $x = \frac{a^2}{c} = \frac{4}{3}$ and $x = -\frac{a^2}{c} = -\frac{4}{3}$

F (3; 0) and the associated directrix is $x = \frac{a^2}{c} = \frac{4}{3}$

3) a- $z' = (1 + i)\, z$ which gives $z = \frac{z'}{1+i}$ then

$$x + iy = \frac{x' + iy'}{1+i} = \frac{(x'+iy')(1-i)}{(1+i)(1-i)} = \frac{x'+y'+i(y'-x')}{2}$$

$$x = \frac{x'+y'}{2} \ and \ \ y = \frac{y'-x'}{2}$$

Replacing x and y by their values in (H) we get:

$$\frac{(x'+y')^2}{16} - \frac{(y'-x')^2}{20} = 1 \ \text{ then } \ x'^2 + y'^2 + 18x'y' = 80$$

The image of (H) is the curve (H′) of equation:

$$x^2 + y^2 + 18xy = 80$$

b- $NK = \dfrac{|3x+3y-8|}{\sqrt{9+9}} = \dfrac{|3x+3y-8|}{3\sqrt{2}}$

$NF_1^{\,2} = (x-3)^2 + (y-3)^2$

$4NF_1^{\,2} = 9NK^2$; $x^2 + y^2 + 18xy = 80$ then the points N vary on the curve (H′)

c- $\dfrac{NF_1}{NK} = \dfrac{3}{2}$, then N describes the hyperbola of focus F_1 , directrix (Δ) and eccentricity $e = \dfrac{3}{2} > 1$

Then (H′) is the hyperbola of focus F_1, of directrix (Δ) and eccentricity $e = \dfrac{3}{2}$