Entrance exam 2002-2003 **Mathematics** **Duration: 3 hours**
July 2002

Remarks: The use of the non-programmable calculator is allowed.
The distribution of the grades is over 25

I- (9 points) The parts A and B are independent.

Suppose that the plane of coordinates is referred to an orthonormal system $(O\,;\vec{i}\,,\,\vec{j}\,)$.

A- Consider the function f defined on $]-\infty, 0[\cup]0,+\infty[$, by $f(x) = x - \ln|x|$ and designate by (C) its representative curve.

1) a- Show that $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to-\infty} f(x) = -\infty$ *and* $\lim_{x\to 0} f(x) = +\infty$

b- Deduce an asymptote to (C)

2) Verify that $f'(x) = \dfrac{x-1}{x}$. Draw a table of variation of f.

3) a- Prove that the straight (d) of equation $y = x$ is an asymptotic direction for (C)

b- Study the relative position of (C) and (d)

4) Draw (d) and (C) in the same system of coordinates.

5) a- Prove that the equation $x = \ln|x|$ admits only one root.

b- Let α be the root. Verify that -0,568< α < -0,566. Take α = -0,567.

B-1) Consider the differential equation (E) : $y'' + 2y' + y = -2e^{-x}$. Let $y = ze^{-x}$

a- Prove that z verifies a differential equation (E´). Solve (E´)

b- Determine the general solution of (E). Deduce the particular solution of (E) verifying y(0) = -1 and y´(0) = -1

2) Consider the function g, defined, on IR, by $g(x) = -(x+1)^2 e^{-x}$ and designate by (C´) its representative curve.

a- Calculate $\lim_{x\to-\infty} g(x)$ and $\lim_{x\to+\infty} g(x)$ Deduce an asymptote to (C´)

b-Verify that $g'(x) = (x^2 - 1)\; e^{-x}$ and deduce the table of variation of g. Draw (C´).

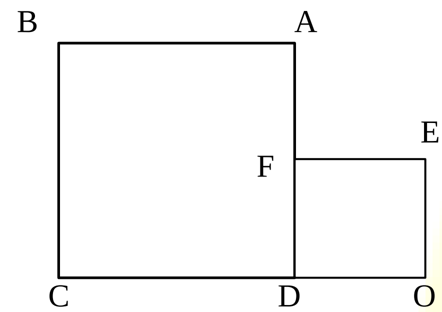
c- Calculate the area of the domain limited by (C´), the x- axis and the two straight lines equations x = -1 and x = 1. (You can find a primitive G of g at the form $G(x) = (ax^2 + bx + c)e^{-x}$).

d- By a graphical reading, indicate, according to the values of the real number k, the number of solutions of the equation g(x) = k.

C- Prove that the equation $g(x) = \ln|g(x)|$ admits 3 roots whose one of them β is positive. Give an inclusion (framing) of β of amplitude of 10^{-2}.

II- (8 points) the parts A, B and C of the problem are independent

In the oriented plane (P), consider the figure drawn aside, formed by the two squares DABC and OEFD so that DA = L. and DF = l .Designate by ***r*** the rotation of center D which transform A in C and by ***s*** the direct similitude of center D, of ratio $\frac{\sqrt{2}}{2}$ and of angle $+\frac{\pi}{4}$



A-1) a- Precise the angle of r and determine r (O).
b-Determine the angle of the straight lines (AO) and (CF).

2) a- Determine s (B) and s (E) .
b- Determine the angle $(\overrightarrow{BE}, \overrightarrow{CF})$.
3) a- Prove that the point **I,** intersection (CF) and (BE) is a point of circum circle of the square ABCD.
b- Deduce the nature of triangle AIC.
c- Prove that the straight lines (AO), (BE) and (CF) are concurrent.

B- Consider the points A_1, A_2, A_3 , …..A_i,…..defined by; $A_1 = s(A)$, $A_2 = s(A_1)$, $A_3 = s(A_2)$,…. , $A_i = s(A_{i-1})$… Let $l_1 = AA_1$, $l_2 = A_1A_2$, $l_3 = A_2A_3$, …. $l_i = A_{i-1}A_i$,…
1) Locate, on the figure, the points A_1 , A_2 , A_3 , A_4.
2) Show that the sequence of general term l_i is a geometric sequence whose first term and common ratio are to be determined.
3) Calculate l_i in terms of L and i.
4) Show that A_8 belongs to [DA].
5) Find a relation between L and l for A_8 to be coinciding with F.

C- Suppose (P) is referred to an orthonormal system $(O ; \vec{u} , \vec{v})$ so that $\vec{u} = -\overrightarrow{OD}$ and $\vec{v} = \overrightarrow{OE}$
1) Write the complex forms of the rotation r and the similitude s
2) Suppose L = 3. Designate by (γ) the circle circumscribed about the square OEFD and by (γ´) the circum circle of DABC.
a- Determine the affix of the center w of the positive homothety h which transforms (γ) into (γ´).
b- Write the complex form of h.

III- (3 points) In a school there are 3 classes of twelve grades: The class T_1 of 20 students containing 8 girls, the class T_2 of 30 students containing 10 girls and the class T_3 of 40 students containing 24 girls. We choose at random a class and from this class we choose at random a student to represent the school during the celebration of the end of the year.

1) What is the probability of the event A : « the chosen student is a girl of T_3 » ?
2) What is the probability of the event B : « the chosen student is a girl » ?
3) Knowing that the chosen student is a girl. What is the probability that this student is from T_3 ?

IV- (5 points) Suppose the space is referred to an orthonormal system $(O ; \vec{i}\ ,\ \vec{j}\ , \vec{k})$ and designate by (P) the plane of equation $4x-3z+12=0$

Let M $(X,Y,0)$ a point of plane (xOy).

1- Calculate, in tems of X, the distance MH of the point M to the plane (P) and show that if MO = MH, then X and Y must verify the relation $9X^2+25Y^2-96X-144=0$

2- Deduce from this relation that the set of points, of the plane (xOy), equidistant from O and (P) is an ellipse (E) whose center, its focii and the eccentricity are to be determined.

3- The plane (P) cuts (xOy) along a straight line (d).
   a- Write a system of parametric equations of (d).
   b- Show that (d) is a directrice of (E).

Entrance Exam 2002-2003 **<u>Solution of Mathematics</u>** **Duration: 3 hours**
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I. Part A

1) a- $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} x\left(1-\dfrac{\ln|x|}{x}\right) = +\infty$

$\lim\limits_{x\to-\infty} f(x) = -\infty - \infty = -\infty$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} -\ln|x| = +\infty$

b- $x = 0$ is an asymptote to (C)

2) $f'(x) = 1-\dfrac{1}{x} = \dfrac{x-1}{x}$, then the table of variations of *f* is the following:

x	$-\infty$		0		1		$+\infty$
$f'(x)$		$+$	‖	$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	$+\infty$ ‖ $+\infty$	$\searrow$	1	$\nearrow$	$+\infty$

3) a- $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to+\infty}\left(1-\dfrac{\ln|x|}{x}\right) = 1$

$\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to-\infty}\left(1-\dfrac{\ln|x|}{x}\right) = 1$

$\lim\limits_{x\to+\infty}\left(f(x)-x\right) = \lim\limits_{x\to+\infty}\left(-\ln|x|\right) = -\infty$

$\lim\limits_{x\to-\infty}\left(f(x)-x\right) = \lim\limits_{x\to-\infty}\left(-\ln|x|\right) = -\infty$

Then, the straight line of equation y = x is an asymptotic direction to (C) at $+\infty$ and at $-\infty$

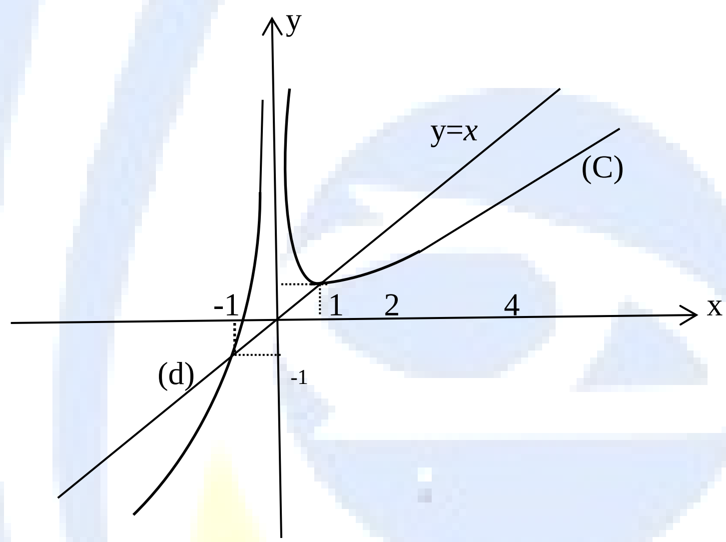
b- $f(x) - x = -\ln|x|$ then the following 3 cases result :

(d) is above (C) for $|x|<1$ that is for -1< x <1 with $x \neq 0$.

(d) is below (C) for $x > 1$ or $x < -1$.

(d) cuts (C) at the points of abscissas (1;1) and (-1; -1).

4)



5) a- The equation $x = \ln|x|$ is equivalent to $x - \ln|x| = 0$ that is to $f(x) = 0$, so it is sufficient to study the intersection between (C) and the axis $x' x$.

Graphically, we notice that (C) cuts x' at a unique point of negative abscissa.

b- : f(-0,568) = -0,002<0 and f(-0,566) = 0,0031>0

then -0,568 < α < -0,566. so α = -0,567

B-1) a- $y' = e^{-x}(z' - z)$ and $y'' = e^{-x}(z'' - 2z' + z)$

Replacing y' and y'' in (E) we get: $z'' = -2$ (E′).

Which gives $z' = -2x + a$ and $z = -x^2 + a x + b$

b- The general solution (E) is $y = (-x^2 + a x + b)\, e^{-x}$

at $y(0) = -1$ gives $b = -1$.

$y' = e^{-x}(-2x + a + x^2 - ax - b)$, $y'(0) = -1$ gives $a - b = -1$ then $a = -2$ and consequently the particular solution of (E) is : $y = (-x^2 - 2x - 1)e^{-x} = -(x+1)^2 e^{-x}$.

2) a- $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty}[-(x+1)^2 e^{-x}] = -\infty$

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty}\left[\frac{-(x+1)^2}{e^x}\right] = 0$$

Then, the axis $x'x$ is an asymptote to (C´).

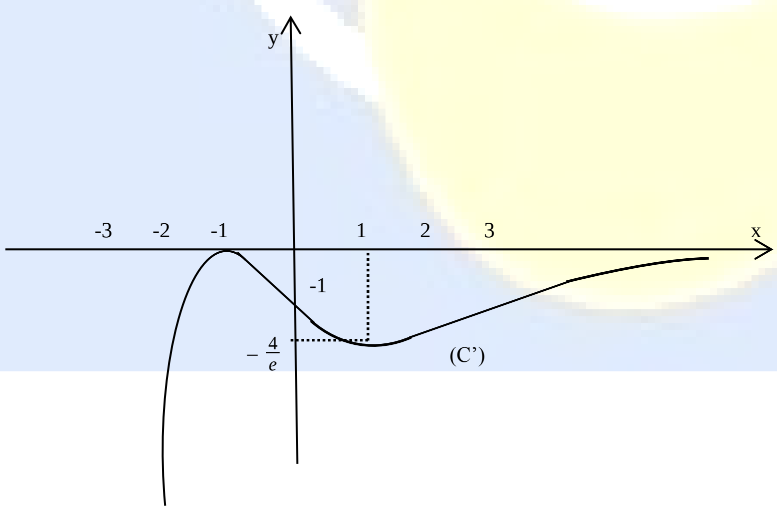
$$\lim_{x\to-\infty}\frac{g(x)}{x} = \lim_{x\to-\infty}\left[-(x+1)^2\frac{e^{-x}}{x}\right] = +\infty$$

Hence, $y'y$ is an asymptotic direction to (C´) at $+\infty$.

b- $g'(x) = -2(x+1)e^{-x} + e^{-x}(x+1)^2 = (x^2-1)e^{-x}$, then, the table of variations of g is:

x	$-\infty$		-1		1		$+\infty$
$g'(x)$		$+$	0	$-$	0	$+$	
$g(x)$	$-\infty$	$\nearrow$	0	$\searrow$	$-\frac{4}{e}$	$\nearrow$	0

$-\frac{4}{e} \approx -1{,}471.$



c- $A = -\int_{-1}^{1} g(x)dx,\ if\ \ G(x) = (ax^2 + bx + c)e^{-x}$ is antiderivative of g we get:

$G'(x)$ = g(x) , which gives: $-ax^2 - (b-2a)x - (-b+c) = -x^2 - 2x - 1,$

so a = 1 ; b = 4 ; c = 5 and consequently G(x) = (x^2 + 4x +5) e^{-x}

Therefore, $A = -[G(x)]_{-1}^{1} = G(-1) - G(1) = 2e - \frac{10}{e}u^2$

d- Let (d) be the straight line of equation $y = k$, parallel to $x'x$

If k < $-\frac{4}{e}$; (d) cuts (C) in a unique point, then the equation has one root only.

If k = $-\frac{4}{e}$, there are two roots which one is a double root x = 1

If $-\frac{4}{e}$ < k < 0 there are three roots.

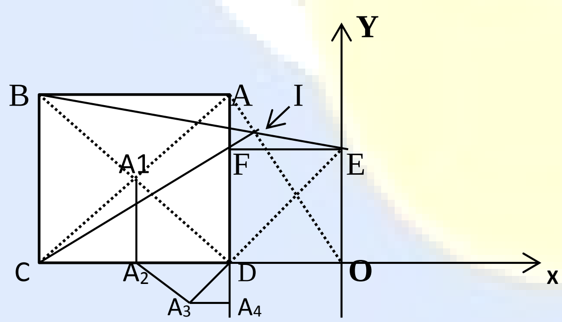
If k = 0, there is one double root x = -1

If k > 0, there are no roots.

C- The equation $x = \ln|x|$ has one negative root α = - 0,567 then $g(x) = \ln|x|$ has solutions for

g(x) = - 0,567 But -1< - 0,567 < 0 then there 3 values of x of which one only is positive ; $\beta > 1$. And since g(3,63) = - 0,568 and g(3,64) = - 0,565, we deduce that 3,63 < β < 3,64.

II-



A- 1) a- $(\overrightarrow{DA}, \overrightarrow{DC}) = \frac{\pi}{2}(2\pi)$ and $DA = DC$ then $r = rot\left(D, \frac{\pi}{2}\right)$.

r (O) = F, since $DO = DF$ and $(\vec{DO}, \vec{DF}) = \frac{\pi}{2}(2\pi)$.

b- r (A) = C and $r(O)= F$, then, $(\vec{AO}, \vec{CF}) = \frac{\pi}{2}(2\pi)$ so the two straight lines (AO) and (CF) are perpendicular.

2) a- we have $\begin{cases} DC = \frac{\sqrt{2}}{2} DB \\ (\vec{DB}, \vec{DC}) = \frac{\pi}{4}(\text{mod} 2\pi) \end{cases}$ and $\begin{cases} DF = \frac{\sqrt{2}}{2} DE \\ (\vec{DE}, \vec{DF}) = \frac{\pi}{4}(2\pi) \end{cases}$

then $s(B) = C$ and $s(E) = F$

b- $s(B) = C$ and $s(E) = F$, then $(\vec{BE}, \vec{CF}) = \frac{\pi}{4}(2\pi)$

3) a- $(\vec{IB}, \vec{IC}) = (\vec{AB}, \vec{AC}) = \frac{\pi}{4}(2\pi)$ then the points I, A, B and C belong to the same circle then I is a point of the circle circumscribed about square $ABCD$

b-[AC] is a diameter of the circle circumscribe about square $ABCD$ then $\widehat{AIC}$ = 90º and consequently triangle AIC is right at I.

c- (AI) is perpendicular to (CF) and (AO) is perpendicular to (CF) then A, I, O are collinear, consequently, (AO), (BE) and (CF) are concurrent at I.

B- 1) a- A_1 = s(A), then A_1 is the midpoint of [BD]

$A_2 = s(A_1)$, then triangle DA_1A_2 is right isosceles and A_2 Є (CD).

$A_3 = s(A_2)$, then triangle DA_2A_3 is right isosceles and A_3 Є (DE).

$A_4 = s(A_3)$, then triangle DA_3A_4 is right isosceles and A_4 Є (DA) ;

2) $\ell_i = A_{i-1}A_i$, $then\, \ell_i = AA_1 = \frac{\sqrt{2}}{2} DA = \frac{\sqrt{2}}{2} L$

Triangle $DA_{i-1}A_i$ is right isosceles at A_i , then :

$\ell_i = \frac{\sqrt{2}}{2} DA_{i-1} = \frac{\sqrt{2}}{2} A_{i-2}A_{i-1} = \frac{\sqrt{2}}{2} \ell_{i-1}$ so (ℓ_i) is a geometric sequence of the first term

$\ell_1 = \frac{\sqrt{2}}{2}L$ and of common ratio $\frac{\sqrt{2}}{2}$.

3) $\ell_i = \ell_1\left(\frac{\sqrt{2}}{2}\right)^{i-1} = \frac{\sqrt{2}}{2}L\left(\frac{\sqrt{2}}{2}\right)^{i-1} = L\left(\frac{\sqrt{2}}{2}\right)^{i}$

4) $(\overrightarrow{DA}, \overrightarrow{DA_8}) = 8\times\frac{\pi}{4}(2\pi) = 0(2\pi)$ *and* $DA_8 < L$. Therefore, $A_8 \in [DA]$.

5) $A_8 \equiv F$ gives $A_7A_8 = \ell_8 = DA_8 = DF$ Therefore: $\ell = \ell_8 = L\left(\frac{\sqrt{2}}{2}\right)^8$ *so* $\ell = \frac{L}{16}$

C -1) D (-1 ; 0), the complex form of r is :

$r:\ z \to Z = az+b$, *with* $a = i$ *and* $-1 = \frac{b}{1-i}$; $b = -1+i$ therefore $Z = iz + i - 1$.

The complex form of s is : s: $z \to Z = az+b$ with : $a = \frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}} = \frac{1+i}{2}$ and

$-1 = \frac{b}{1-\frac{1}{2}-\frac{i}{2}}$ *so* $b = \frac{i}{2}-\frac{1}{2}$ therefore $Z = \left(\frac{1+i}{2}\right)z - \frac{1}{2}(1-i)$

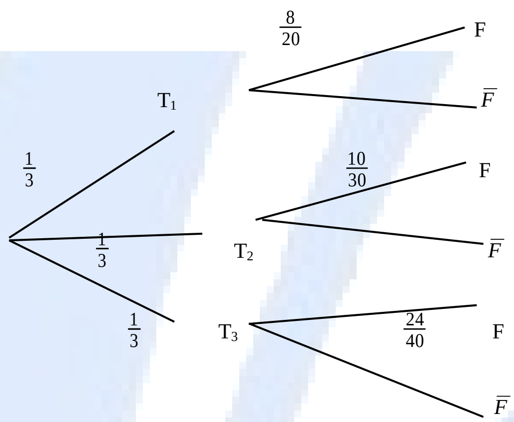
2) a- The ratio of the dilation is k = 3. The center of (γ) is $I\left(-\frac{1}{2},\frac{1}{2}\right)$ and the center of (γ') is $I'\left(-\frac{5}{2},\frac{3}{2}\right)$.

$h(I) = I'$ then $\overrightarrow{WI'} = 3\overrightarrow{WI}$ then $z_{r} - z_w = 3z_1 - 3z_w$ which gives

$2z_w = 3z_1 - 3z_r = 1$, *so* $z_w = \frac{1}{2}$

b- The complex form of h is : h: $z \to Z = kz+b$ *with* $k = 3$ *and* $\frac{1}{2} = \frac{b}{1-3}$ therefore b = -1 and consequently $Z = 3z - 1$.

III- 1) $p(A) = p(F \cap T_3) = p(T_3).p(F/T_3) = \frac{1}{3} \times \frac{24}{40} = \frac{1}{5}$



2) $p(B) = p(F) = p(F \cap T_1) + p(F \cap T_2) + p(F \cap T_3) = \frac{4}{9}$

$$= \frac{1}{3} \times \frac{8}{20} + \frac{1}{3} \times \frac{10}{30} + \frac{1}{3} \times \frac{24}{40} = \frac{4}{9}$$

3) $p(T_3/F) = \frac{p(T_3 \cap F)}{p(F)} = \frac{p(A)}{p(B)} = \frac{9}{20}$

IV- 1) $MH = \frac{|4x+12|}{\sqrt{16+9}} = \frac{4}{5}|x+3|$.

MO =MH gives $\overline{MO}^2 = \overline{MH}^2$ then:

$x^2 + y^2 = \frac{16}{25}(x^2 + 9 + 6x) = 0$ that is $9x^2 + 25y^2 - 96x - 144 = 0$

2) The set of points of the plane (XOY) equidistant from O and from (P) is the curve of equation

$9x^2 + 25y^2 - 96x - 144 = 0$ which is equivalent to : $\frac{\left(x - \frac{16}{3}\right)^2}{\frac{400}{9}} + \frac{y^2}{16} = 1$

It is an ellipse of center $w(\frac{16}{3},0)$ and focal axis $x' x$.

But $c^2 = a^2 - b^2 = \frac{400}{9} - 16 = \frac{256}{9}$ $then\, c = \frac{16}{3}$.

The foci are : $F(\frac{16}{3}+\frac{16}{3},0) \equiv F(\frac{32}{3},0)$ and $F'\left(\frac{16}{3}-\frac{16}{3},0\right) \equiv F'(0,0)$ and $e = \frac{c}{a} = \frac{\frac{16}{3}}{\frac{20}{3}} = \frac{4}{5}$

3) (d) is the intersection of (P) and (xOy), then : (d) $\begin{cases} 4x - 3z + 12 = 0 \\ z = 0 \end{cases}$

a-A system of parametric equations of (d) is: (d) $\begin{cases} x = -3 \\ y = m \\ z = 0 \end{cases}$

b- One of the foci is *0*

$MO = MH = \frac{4}{5}|x+3|$ and $d(M;(d)) = |x+3|$, $therefore$ $\frac{MO}{d(M,(d))} = \frac{\frac{4}{5}|x+3|}{|x+3|} = \frac{4}{5} = e$

Hence, (d) is the directrix of (E) associated to the focus O.