LEBANESE UNIVERSITY
FACULTY OF ENGINEERING

الجامعة اللبنانية
كلية الهندسة

Entrance Exam 2001 - 2002 **Mathematics** **Duration : 3 hours**
July 2001

Remark: The use of a calculator with no programs is allowed.
The distribution of grades is over 25

I- (2 points)

Solve the inequation $\ln\left(\frac{x+1}{5-x}\right) > \ln(2x-3)$

II- (5points)

Suppose the complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$. Designate by A the point of affix 1, by B the point of affix i , by (C) the circle of center O and of radius 1 and by (D) the straight line of equation y = 1.

To every point M of affix z $\neq$ i, we associate the point M' of affix $z' = \frac{z-i}{\bar{z}+i}$

1) Determine the set of M of affix z so that z´ = 1.
2) Show that $z'\,\bar{z}' = 1$. Interpret geometrically the result.
3) a- Show that, for every M which does not belong to (D), $\frac{z'-1}{z-i}$ is pure imaginary.
   b- Prove that the two straight lines (AM') and (BM) are perpendicular.
   c- M being a given point which does not belong to (D), construct geometrically point M´.
   d- Precise the position of M' when M belongs to (D) deprived from B.

III- (5 points)

In the complex plane referred to an orthonormal system $(O;\vec{u},\vec{v})$. Consider the points: A_0 , A_1 ,…., A_n , A_{n+1} ,..., of respective affixes z_0 , z_1 , …, z_n , z_{n+1} , …, defined by :

$z_0 = 0$ and $z_{n+1} = \frac{1}{1+i} z_n + i$ $\quad (n \in \mathbb{N})$.

1) Show that, whatever is n, A_{n+1} is the image of A_n by a direct similitude whose center I, its ratio k and its angle α are to be determined.

2) a- Prove that, whatever is n, the triangle $IA_n A_{n+1}$ is right angled at A_{n+1} .
   b- Deduce a construction of A_{n+1} using A_n and place the A_0 , A_1 , A_2 , A_3 , A_4 , A_5 (for drawing the figure and only for this purpose, take as unit of length 4 cm).

3) Suppose a_k = area ($IA_k \; A_{k+1}$) and $S_n = a_0 + a_1 + a_2 + \ldots + a_n$
   a- Show that the sequence of general term a_k is a geometric sequence whose first term and its common ratio are to be determined.
   b- Calculate S_n in terms of n and determine its limit when n tends to $+\infty$.

IV- (4 points)

We consider 3 urns U_1 , U_2 and U_3 , containing each 6 balls :
U_1 contains 2 blue balls and 4 red ones.
U_2 contains 3 blue balls and 3 red ones.
U_3 contains 5 blue balls and 1 red ball.

1) *In this part, consider the urn* U_1 . We draw from it a ball at random.
   This operation is repeated 5 times replacing the ball each time in the urn U_1 .
   a- What is the probability to obtain 4 blue balls and 1 red ball in the following order : blue, blue, blue, blue, red ?
   b- What is the probability to obtain 4 blue balls 1 red ball in any order ?
   c- What is the probability to obtain at least one blue ball ?

2) *In this part, we choose at random an urn from the three urns* U_1 , U_2 , U_3 , and we draw at random from it a ball.
   a- What is the probability to obtain a blue ball?
   b- We know the draw ball is blue; what is the probability that the ball comes from U_3 ?

V- (9 points)

A- Consider the function f , defined over $]0, +\infty[$, by $f(x) = \dfrac{x-1}{x}(\ln x - 2)$

and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $\lim\limits_{x\to+\infty} f(x) = +\infty$ and $\lim\limits_{x\to 0} f(x) = +\infty$.

2) Show that f is differentiable over $]0, +\infty[$ and that $f'(x) = \dfrac{1}{x^2}(\ln x + x - 3)$

3) Let g be the function defined over $]0, +\infty[$ by $g(x) = \ln x + x - 3$
   a- Study the variation of g .
   b- Show that $g(x) = 0$ has a unique solution α and that $2.20 < \alpha < 2.21$.
   c- Study the sign of $g(x)$ over $]0, +\infty[$.

4) a- Study the variation of f.

b- Show that $f(\alpha) = -\dfrac{(\alpha-1)^2}{\alpha}$. Deduce that $-0.67 \le f(\alpha) \le -0.65$

5) a- Study the sign of f (x) and show that f (x) < 0 if and only if $x \in]1, e^2[$.

b- Calculate f (1) and f (e^2) and draw (C).

B- Consider the function F defined over $]0, +\infty[$ by $F(x) = \int_1^x f(t)dt$. We call (Γ) the representative curve of F .

1) a- Without calculating F (x), study the variations of F over $]0, +\infty[$.

b- What can be said about the tangents to (Γ) at its points of abscissas 1 and e^2 ?

2) a- Prove that $\int_1^x \ln(t)dt = x\ln x - x + 1$

b- Prove that $F(x) = x\ln x - 3x - \frac{1}{2}(\ln x)^2 + 2\ln x + 3$

c- Calculate $\lim\limits_{x\to 0} F(x)$.

d- Noticing that $F(x) = x\ln x\left(1 - \dfrac{3}{\ln x} - \dfrac{1}{2}\dfrac{\ln x}{x} + \dfrac{2}{x}\right) + 3$, calculate

$\lim\limits_{x\to+\infty} F(x)$ and $\lim\limits_{x\to+\infty} \dfrac{F(x)}{x}$

e- Set up a table of variations of F and draw (Γ) .

3) Calculate the area S of the domain limited by (C), the axis of abscissas and the two straight lines of equations x = 1 and $x = e^2$. Give an approximated value of S to the nearest 10^{-3} by greater value.

Entrance exam 2001-2002 **Solution of Mathematics** **Duration: 3 hours**
July 2001

I- This inequality is defined for: $\begin{cases} \dfrac{x+1}{5-x} > 0 \\ 2x-3>0 \end{cases}$

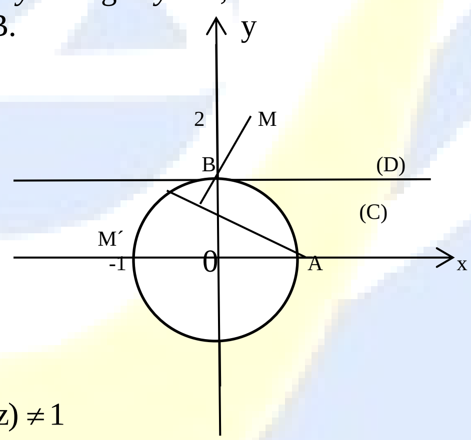
Which gives -1< x < 5 and $x > \dfrac{3}{2}$, that is for $\dfrac{3}{2} < x < 5$.

The inequality: $\ln\left(\dfrac{x+1}{5-x}\right) > \ln(2x-3)$ gives $\dfrac{x+1}{5-x} > 2x\text{-}3$

Therefore $\dfrac{2x^2-12x+16}{5-x} > 0$ which is verified for x < 2 or 4< x< 5

The accepted solution is then: 4 < x < 5 or $\dfrac{3}{2} < x < 2$

II- 1) z´=1 gives $z-i=\bar z+i$, then z - $\bar z$ = 2i and if z = x + $i\,y$ we get y = 1, then the set of points M is the straight line (D) deprived of the point B.



2) z´ $\bar z$ ´ = $\dfrac{z-i}{\bar z+i}\times\dfrac{\bar z+i}{z-i}=1$

But $z'\,\bar z' = |z'|^2 = 1 = OM'^2$ *so* $OM' = 1$

And consequently the point M' is a point of circle (C)

3) a- M does not belong to (D), so Im (z) $\neq$ 1.

$$\frac{z'-1}{z-i} = \frac{\dfrac{z-i}{\bar z+i}-1}{z-i} = \frac{z-i-\bar z-i}{(z-i)\overline{(z-i)}} = \frac{2i(\mathrm{Im}(z)-1)}{|z-i|^2} \text{ and } \mathrm{Im}(z)\neq 1$$

Hence $\dfrac{z'-1}{z-i}$ is pure imaginary.

b- $\dfrac{z_{\overrightarrow{AM'}}}{z_{\overrightarrow{BM}}} = \dfrac{z_{M'}-z_A}{z_M-z_B} = \dfrac{z'-1}{z-i}$ that is pure imaginary, so the two straight lines (AM') and (BM) are perpendicular.

c- M does not belong to (D), then (AM') and (BM) are perpendicular, so M' is on the perpendicular through A to mathematics solution
(BM) and M' is a point of (C), so M' is a point of intersection of these two sets other than A.

d- If M is a point of (D) deprived of B then $z = x + i$ with $x \neq 0$

$z' = \dfrac{x+i-i}{x-i+i} = 1$ So M' is confounded with A.

III- 1) $z_{n+1} = \dfrac{1}{1+i} z_n + i = \dfrac{1-i}{2} z_n + i$ **,** which is the complex form of a similitude.

$a = \dfrac{1-i}{2} = \dfrac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}}$ and $z_1 = \dfrac{b}{1-a} = \dfrac{i}{1-\dfrac{1}{1+i}} = 1+i$

So A_{n+1} is the image of A_n by the direct similitude of center I ($1+i$), ratio $\dfrac{\sqrt{2}}{2}$

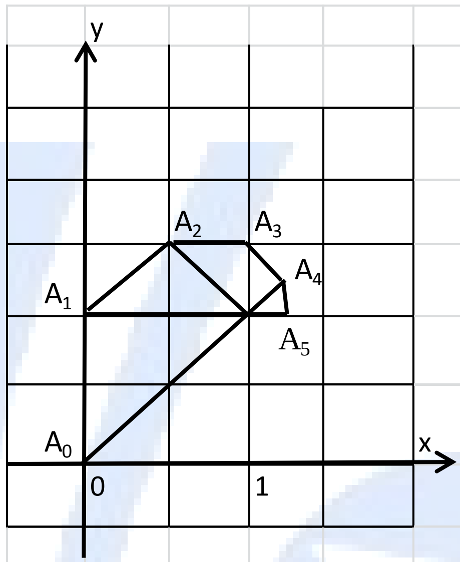
and angle $-\dfrac{\pi}{4}$.

2) a- $(\overrightarrow{IA_n}; \overrightarrow{IA_{n+1}}) = \dfrac{-\pi}{4} (2\pi)$ et $IA_{n+1} = \dfrac{\sqrt{2}}{2} IA_n$, so triangle IA_nA_{n+1} is right at A_{n+1}

b- IA_nA_{n+1} is right isosceles of principal vertex A_{n+1} and

$(\overrightarrow{A_{n+1}I}; \overrightarrow{A_{n+1}A_n}) = \dfrac{-\pi}{2} (2\pi)$ *then* A_{n+1} is the intersection of the semi-circle of diameter [IA_n] and of the perpendicular bisector of [IA_n].

$z_0 = 0$, $z_1 = i$, $z_2 = \dfrac{1}{2} + \dfrac{3}{2} i$, $z_3 = 1 + \dfrac{3}{2} i$, $z_4 = \dfrac{5}{4} + \dfrac{5}{4} i$, $z_5 = \dfrac{5}{4} + i$



3) a- $a_k = \text{Area of } (IA_k A_{k+1}) = \frac{1}{2} IA_k \times IA_{k+1} \times \sin\left(\frac{\pi}{4}\right) = \frac{1}{4} IA_k^{\ 2}$

$$= \frac{1}{4}\left(\frac{\sqrt{2}}{2} IA_{k-1}\right)^2 = \frac{1}{4} \times \frac{1}{2} \times IA^2{}_{k-1} = \frac{1}{2} a_{k-1}$$

Then, a_k is the general term of geometric sequence of initial term $a_0 = \frac{1}{4} IA_0^{\ 2} = \frac{1}{2}$

and of common ratio $r = \frac{1}{2}$

b- $S_n = a_0 \frac{1-q^{n+1}}{1-q} = \frac{1}{2} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^{n+1}$

$\lim\limits_{n \to +\infty} S_n = 1$

IV-1) a- *p* (of getting *B, B, B, B, R)* = $\frac{2}{6} \times \frac{2}{6} \times \frac{2}{6} \times \frac{2}{6} \times \frac{4}{6} = \frac{2}{243}$

b- To get a red ball from the 5 drawn balls is to get:

(R, B, B, B, B) or (B, R, B, B, B) or (B, B, R, B, B) or (B, B, B, R, B) or (B, B, B, B, R)

Then p (of getting *1R and 4B)*= $5 \times \frac{2}{243} = \frac{10}{243}$

c- The event : getting at least one blue ball is the opposite of the event: the 5 balls are red, then :

p (at least one blue ball)= $1-\left(\frac{4}{6}\right)^5 = \frac{211}{243}$

2) a- Designating by B the event : the drawn ball is blue

U_i : the ball comes from U_i

$$p(B) = p(U_1)\times p(B/U_1) + p(U_2)\times p(B/U_2) + p(U_3)\times p(B/U_3)$$

$$p(B) = \frac{1}{3}\times\frac{2}{6} + \frac{1}{3}\times\frac{3}{6} + \frac{1}{3}\times\frac{5}{6} = \frac{5}{9}$$

b- $p(U_3/B) = \dfrac{p(U_3 \cap B)}{p(B)} = \dfrac{\frac{5}{18}}{\frac{5}{9}} = \dfrac{1}{2}$

V- A. 1) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left(\frac{x-1}{x}\right)\times\lim\limits_{x\to+\infty}(\ln x - 2) = 1\times(+\infty) = +\infty$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\left(\frac{x-1}{x}\right)\times\lim\limits_{x\to 0^+}(\ln x - 2) = (-\infty)\times(-\infty) = +\infty$

2) f is differentiable over $]0\,;+\infty[$ since it is the product of two differentiable functions on $]0\,;+\infty[$

$$f'(x) = \frac{x-x+1}{x^2}(\ln x - 2) + \frac{x-1}{x}\times\frac{1}{x} = \frac{1}{x^2}(\ln x + x - 3)$$

3) a- $g'(x) = \frac{1}{x} + 1 > 0$ for all $x \in]0;+\infty[$, then, g is strictly increasing over $]0;+\infty[$

b- $\lim\limits_{x\to 0} g(x) = -\infty$ *and* $\lim\limits_{x\to+\infty} g(x) = +\infty$

g is continuous and strictly increasing and g(x) varies from $-\infty$ to $+\infty$ so the equation g(x) = 0 has one unique solution α .

g (2, 20) = ln (2, 20) + 2,20 – 3 $\approx$ - 0,01 < 0

g (2, 21) = ln (2, 21) + 2,21 – 3 $\approx$ 0,002 >0

Then 2, 20 < α < 2, 21

c- g (x) < 0 for $0 < x < \alpha$ and g (x) > 0 for $x > \alpha$

g (x) = 0 for $x = \alpha$

4) a- $f'(x)$ has the same sign as g(x), then the table of variations of f is :

x	0		α		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$f(\alpha)$	$\nearrow$	$+\infty$

b- $f(\alpha)=\dfrac{\alpha-1}{\alpha}(\ln\alpha-2)$ and $\ln\alpha+\alpha-3=0$

then $f(\alpha)=\dfrac{\alpha-1}{\alpha}(-\alpha+3-2)=\dfrac{-(\alpha-1)^2}{\alpha}$,

$2{,}20<\alpha<2{,}21$ then $1{,}20<\alpha-1<1{,}21$

and $1{,}44<(\alpha-1)^2<1{,}4641$

therefore

$\dfrac{1}{2{,}21}<\dfrac{1}{\alpha}<\dfrac{1}{2{,}20}$ then $\dfrac{1{,}44}{2{,}21}<\dfrac{(\alpha-1)^2}{\alpha}<\dfrac{1{,}4641}{2{,}20}$

and consequently $-0{,}67<f(\alpha)<-0{,}65$

5) a- $f(x)=0$ gives $\left(\dfrac{x-1}{x}\right)(\ln x-2)=0$ so $x=1$ or $x=e^2$

Then the curve (C) cuts the axis $x'x$ at two points of abscissas 1 and e^2 and since $f(\alpha)<0$ then $0<1<\alpha$ and $e^2>\alpha$

x	0		1		α		e^2		$+\infty$
$f(x)$	$\Vert$	$+$	0	$-$	$f(\alpha)$	$-$	0	$+$	

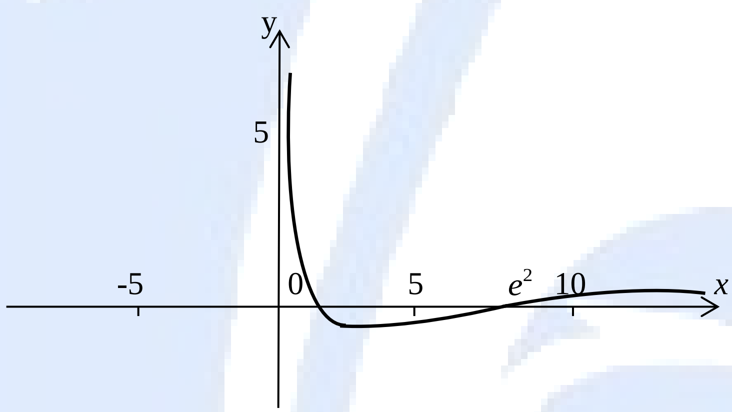
Then $f(x)>0$ for $0<x<1$ or $x>e^2$

$f(x)<0$ for $1<x<e^2$ then $f(x)<0$ if and only if $x\in]1;e^2[$

b- $f(1)=0$ and $f(e^2)=0$. (we get $\beta=1$ and $\gamma=e^2$)

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\left[\frac{x-1}{x}\left(\frac{\ln x}{x}-\frac{2}{x}\right)\right]=0$$

Hence $x'x$ is an asymptotic direction.



B- 1) a- $F'(x)=f(x)$ with $F(1)=0$, then the table of variations of F is:

x	0		1		e^2		$+\infty$
$F'(x)$	‖	$+$	0	$-$	0	$+$	
$F(x)$	‖	$\nearrow$	0	$\searrow$		$\nearrow$	

b- The tangents at (Γ) at the points of abscissas 1 and e^2 are parallel to $x'x$ since $F'(1)=F'(e^2)=0$

2) a- Taking $u=\ln t$ and $v'=1$, we get : $u'=\frac{1}{t}$ and $v=t$, then,

$$\int_1^x \ln t\,dt = t\ln t\Big|_1^x-\int_1^x dt = x\ln x - x + 1$$

b- $$F(x)=\int_1^x\left(\frac{t-1}{t}\right)(\ln t-2)dt=\int_1^x\left(\ln t-2-\frac{\ln t}{t}+\frac{2}{t}\right)dt$$

$$=\left[t\ln t-t-2t-\frac{1}{2}\ln^2 t+2\ln t\right]_1^x$$

$$= x\ln x - x - 2x - \frac{1}{2}\ln^2 x + 2\ln x - (-3)$$

$$= x\ln x - 3x - \frac{\ln^2 x}{2} + 2\ln x + 3$$

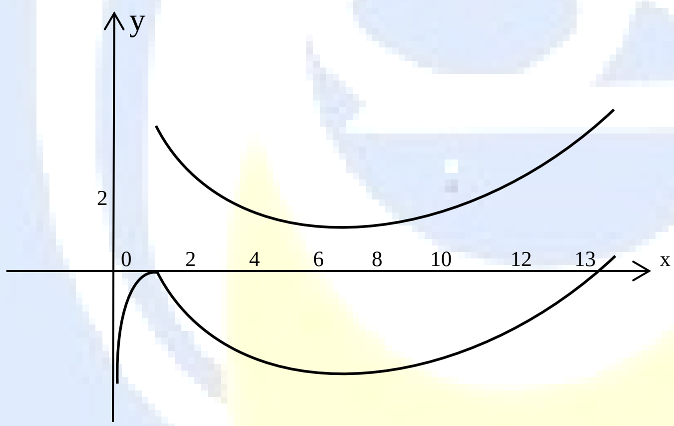
c- $\lim_{x\to 0} F(x) = -\infty$

d- $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} x\ln x(1 - \frac{3}{\ln x} - \frac{1}{2}\frac{\ln x}{x} + \frac{2}{x}) + 3 = +\infty$

$$\lim_{x\to\infty} \frac{F(x)}{x} = +\infty$$

e- $x = 0$ is a vertical asymptote to (Γ). y´y is an asymptotic direction of (Γ) at $+\infty$

$F(1) = 0$, $F(e^2) = 5 - e^2 \approx -2{,}389$



3) For $x \in]1; e^2[$, (Γ) is below $x'x$, then

$$S = -\int_1^{e^2} f(x)dx = -F(e^2) + F(1) = e^2 - 5 \approx 2{,}390\ \text{u}^2 \ .$$