Lebanese University
Faculty of Engineering

Entrance Exam (TEST A)
MATHEMATICS
(Lebanese Program)

Date: *16/07/2022*
Duration: *3h*

Smartphones and notes are strictly prohibited.
Non graphical calculators are allowed.

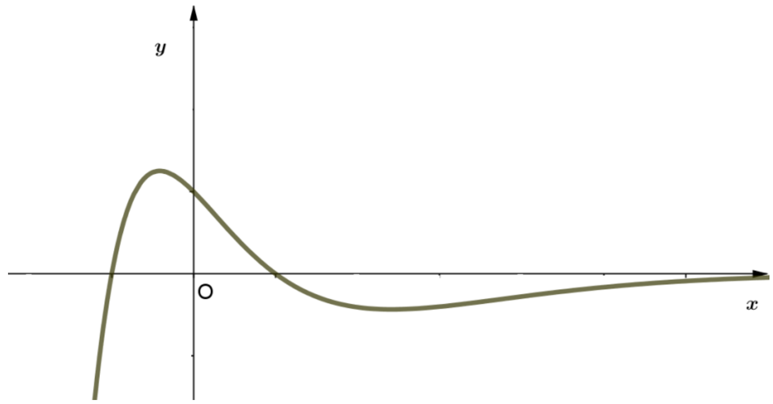
Exercise 1: (14 points)

*This exercise is a multiple-choice problem. For each question, **only one** of the four answers given is correct. A correct answer is worth ½ point. A wrong answer, a multiple answer or the absence of an answer to a question does not earn or deduct points.*
*For this exercise, you have to answer on the answers sheet, by circling for each question **only one** of the letters **a**, **b**, **c** or **d**.*
No justification is required.

1. The solutions in $\mathbb{C}$ of the equation $z^2+\frac{5}{2}=z$ are :

a) $\frac{1}{2}+\frac{3}{2}i$ and $-\frac{1}{2}+\frac{3}{2}i$
b) $\frac{1}{2}+\frac{3}{2}i$ and $-\frac{1}{2}-\frac{3}{2}i$
c) $\frac{1}{2}+\frac{3}{2}i$ and $\frac{1}{2}-\frac{3}{2}i$
d) $-\frac{1}{2}+\frac{3}{2}i$ and $-\frac{1}{2}-\frac{3}{2}i$

2. On the figure below is plotted the graph (C) of the function f defined by $f(x)=(1-x^2)e^{-x}$.
Consider the points A and B of *(C)*, of abscissas 0 and 1, respectively.



a) The tangent lines to the curve *(C)* at A and B are parallel.
b) The function f admits an absolute (global) minimum.
c) The equation $f(x)=1/4$ admits two real solutions.
d) The curve *(C)* admits a unique inflection point.

3. An urn contains 5 green balls and 3 white balls, indistinguishable. Two balls are randomly drawn successively and without replacement from the urn. Consider then the following events :

- V_1 : « the first ball drawn is green » ;
- V_2 : « the second ball drawn is green » ;
- B_1 : « the first ball drawn is white » ;
- B_2 : « the second ball drawn is white ».

a) The probability of the event V_2 is equal to $\frac{25}{64}$.
b) The probability of the event V_2 is equal to $\frac{5}{8}$.
c) The probability of the event V_2 is equal to $\frac{15}{56}$.
d) None of the three previous statements is correct.

4. Consider the equation $\ln(3x+1)+\ln(2x+1)-\ln(x^2)=0$ where the unknown x is a real variable.
The solution set of this equation is:

a) $\mathcal{S}=]-\frac{1}{3}\ ;+\infty[$.
b) $\mathcal{S}=]-\frac{1}{3}\ ;+\infty[-\{0\}$.
c) $\mathcal{S}=\{\frac{-5-\sqrt{5}}{10};\frac{-5+\sqrt{5}}{10}\}$.
d) None of the three previous answers is correct.

5. At the Lycée National high school, a quarter of the students live in Hadath. We choose 2 high school students at random from Lycée National and we assume that the high school enrollment is large enough for this choice to be considered as a draw with replacement. The probability that among the 2 students chosen, none lives in Hadath, is equal to:

a) $\frac{1}{2}$ **b)** $\frac{3}{16}$ **c)** $\frac{9}{16}$ **d)** None of the three previous answers is correct.

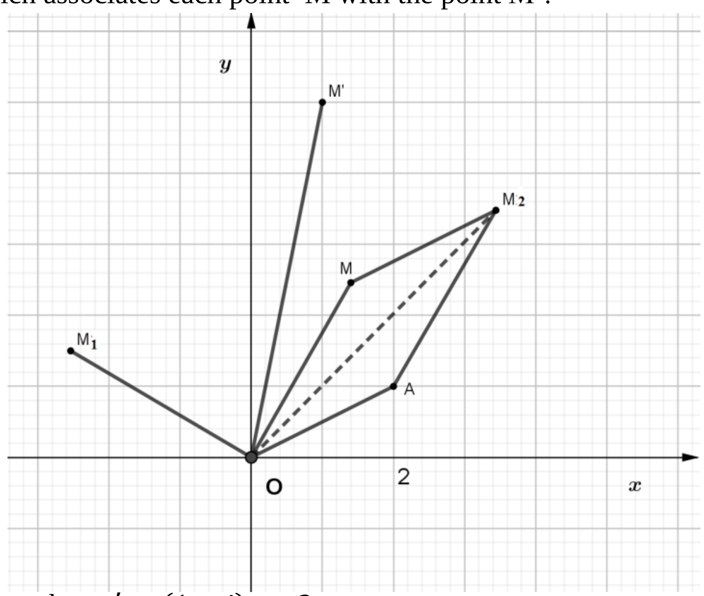
6. A company manufactures smart chips. Each chip can have two defects denoted as A and B. A statistical study shows that 2.8% of the chips have defect A, 2.2% of the chips have defect B, and fortunately 95.4% of the chips have no defect.
The probability that a randomly selected chip has both of the two defects is:

a) 0.046 **b)** 0.004 **c)** 0.904 **d)** None of the three previous answers is correct.

7. It has been found that, over 4 years, the price of a certain commodity has increased by 8% per year.
We can say that, over the 4 years, the price of this commodity has increased, to the nearest unit, by:

a) 12 % **b)** 32 % **c)** 36 % **d)** None of the three previous answers is correct.

8. The plane is referred to a direct orthonormal system $(O, \vec{u}, \vec{v})$. Consider the point A of coordinates (2 ; 1).
Each point M is associated with the point $\mathrm{M_1}$ such that the triangle $\mathrm{MOM_1}$ is right-angled and isosceles at O with $\left(\overrightarrow{\mathrm{OM}}, \overrightarrow{\mathrm{OM_1}}\right) = \frac{\pi}{2}$, and also with the point $\mathrm{M_2}$ such that $\mathrm{OAM_2M}$ is a parallelogram.
Let M′ be the point such that $\overrightarrow{\mathrm{OM'}} = \overrightarrow{\mathrm{OM_1}} + \overrightarrow{\mathrm{OM_2}}$.
Let T be the direct similitude which associates each point M with the point M'.



a) The complex form of T is given by $z' = (1+i)z + 2$.
b) The complex form of T is given by $z' = (1+i)z + 2 + i$.
c) The complex form of T is given by $z' = iz + 2$.
d) None of the three previous statements is correct.

9. The real number $A = e^{\frac{5}{3}} + 3e^3 e^{-\frac{4}{3}}$ is equal to:

a) $e^{\frac{5}{3}} + 3e^{-4}$ **b)** $4 + e^{\frac{5}{3}}$ **c)** $\frac{1}{8}\left(2e^{\frac{1}{3}}\right)^5$ **d)** $-e^{\frac{3}{5}} + \frac{e^9}{e^{\frac{4}{3}}}$

10. Consider the fonction f defined by $f(x) = \ln\left((6x-12)\, e^{\frac{x}{2}} + 2 - x\right)$. Let D_f be the domain of definition of f.

a) $D_f =]-2\ln(6)\,;+\infty[$. **b)** $D_f =]-\infty; -2\ln(6)[\, \cup]2; +\infty[$.
c) $D_f =]2; +\infty[$. **d)** None of the three previous statements is correct.

11. Consider two fair tetrahedral dice. The faces of each die are numbered from 1 to 4.The two dice are rolled. Denote by X the random variable that is defined as follows:

- if the two numbers obtained on the dice are different, then X is equal to the greater between them;
- if the two numbers obtained on the dice are equal, then X is equal to one of them.

The expectation E(X) of the random variable X is equal to:

a) 4 **b)** $\frac{25}{8}$ **c)** 8 **d)** None of the three answers statements is correct.

12. In the complex plane referred to a direct orthonormal system $(O,\vec{u},\vec{v})$, we consider the points $A,\ B$ and H of affices $-5+6i,\ \ 3-2i$ and -5, respectively. Let s be the direct similitude of center A which maps B into H.

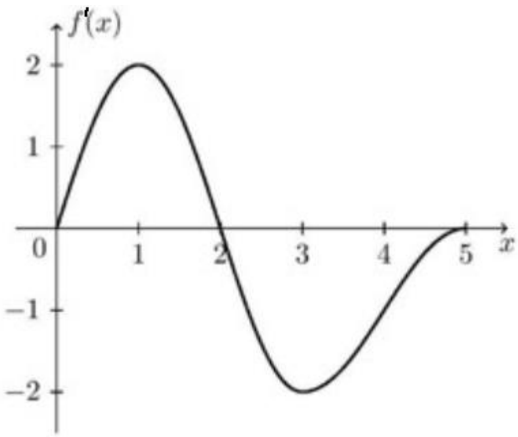
a) s is a homothety (dilation) of ratio ¾ .

b) s is a similitude of scale factor $\frac{3\sqrt{2}}{8}$ and of angle $-\pi/4$.

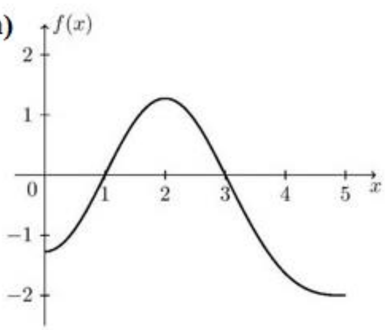
c) s is a similitude of scale factor $\frac{3\sqrt{2}}{8}$ and of angle $-\pi/2$.

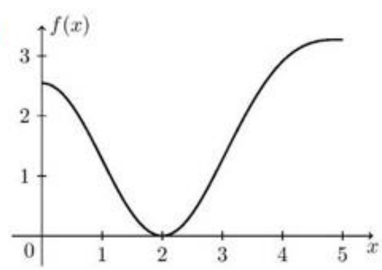
d) None of the three previous statements is correct.

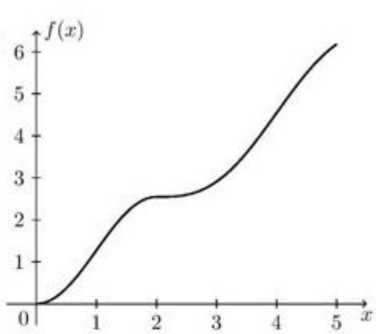
13. Consider a real function f defined and differentiable on $[0;5]$ such that the graph of its derivative f' is represented as follows.

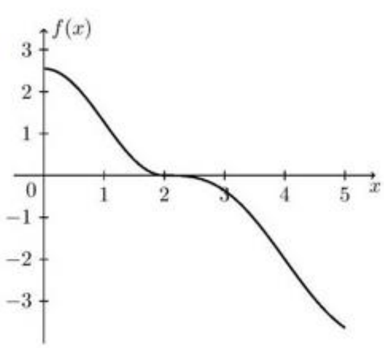


Which one of the following graphs is the graph of f?

a)


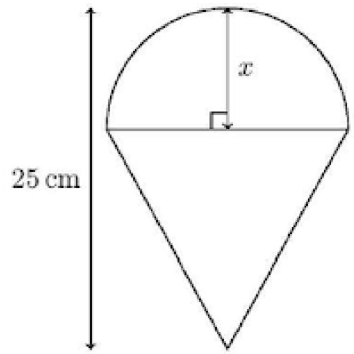
b)


c)


d)


14. I want to sell ice cream. For this, I print posters with a large ice cream cone. The cornet is an equilateral triangle and the ice ball is represented by a semicircle of radius x. In order to have a nice effect, I require that the total height of the ice cream cone plus the ice ball to be 25 cm (see illustration below).
What then is the value of x?

a) $\frac{25}{2}$ **b)** $\frac{25}{1+\sqrt{2}}$ **c)** $\frac{25}{1+\sqrt{3}}$ **d)** $\frac{25(3-\sqrt{3})}{2}$



15. $\lim\limits_{x \to 0^-} x e^{-\frac{1}{x}} =$

a) 0 **b)** $-\infty$ **c)** $+\infty$ **d)** 1

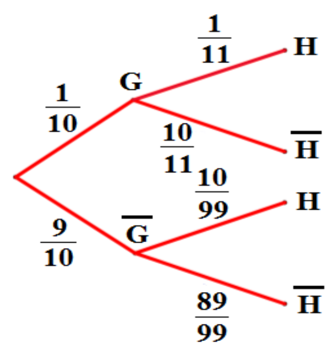
16. Consider the following probability tree.
Then we can assert that :

a) $p(\overline{\mathbf{H}}/\mathbf{G}) = \frac{1}{11}$

b) $p(\mathbf{H}) = \frac{19}{99}$

c) $p(\mathbf{G}/\mathbf{H}) = \frac{1}{11}$

d) $p(\mathbf{G} \cap \overline{\mathbf{H}}) = \frac{10}{11}$



17. Let f be the function defined on $\mathbb{R}$ by $f(x) = \frac{4x}{e^x+1}$, whose representative curve is (C). The equation of the tangent line to (C) at the point of abscissa 0 is given by:

a) $y = 2x + 2$ **b)** $x = \frac{1}{2}y$ **c)** $x = 0$ **d)** $y = 2$

18. In the complex plane referred to a direct orthonormal system $(O, \vec{u}, \vec{v})$, consider the set Ω of points M of affix z such that $|\bar{z} - 1 + i| = |3 - 4i|$.

a) Ω is a straight line.
b) Ω is a circle.
c) $\Omega = \{A, B\}$ where A and B are the points of affices $4 + 5i$ and $-2 - 3i$, respectively.
d) None of the three previous statements is correct.

19. In the complex plane, consider the three points A, B and C of affices a, b and c, respectively.

If $\frac{a-b}{c-b} = -i$, then the triangle ABC is :

a) isosceles and not right-angled. **b)** equilateral. **c)** isosceles and right-angled. **d)** right-angled and not isosceles.

20. Twice the logarithm of a number is equal to the logarithm of half that number. What is this number?

a) -1 **b)** 0 **c)** 0,5 **d)** 2

21. Consider the following real function $f : x \longmapsto \sqrt{\frac{3x}{4x-5}}$.

a) f is defined in $\mathbb{R} - \{\frac{5}{4}\}$.

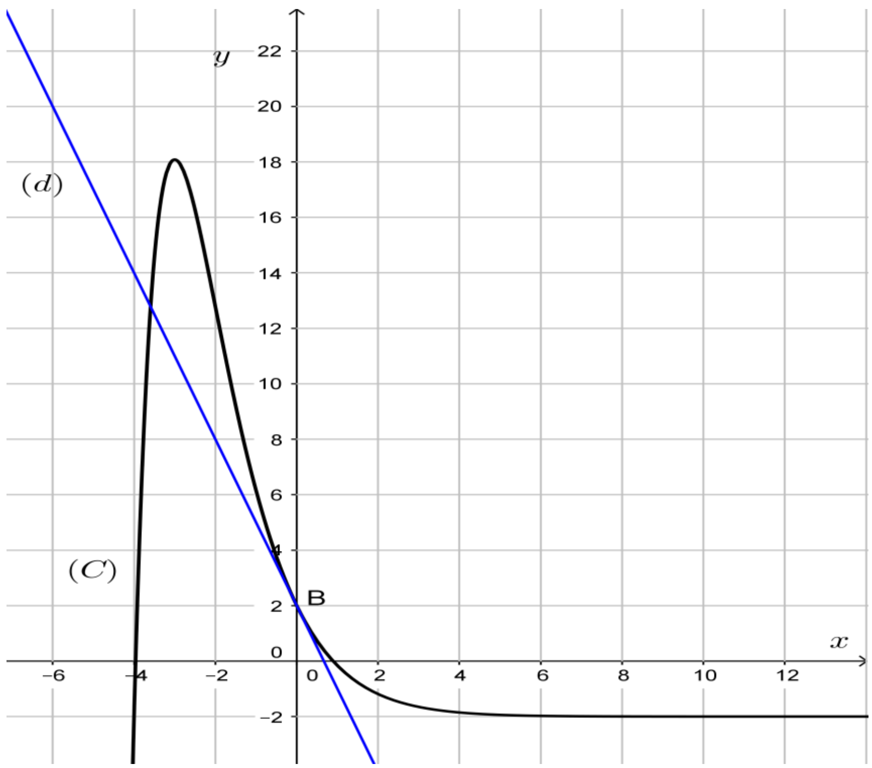
b) f is differentiable in $]-\infty; 0] \cup]\frac{5}{4}; +\infty[$.

c) f is continuous in $]-\infty; 0] \cup [\frac{5}{4}; +\infty[$.

d) None of the three previous statements is correct.

22. The curve (C) below represents, in an orthogonal system, a real function f defined and differentiable on $\mathbb{R}$. The point B(0; 2) belongs to the curve (C). We have also represented on this graph the straight line (d) tangent to the curve (C) at point B. The straight line (d) passes through the point of coordinates (-2; 8).

The function f is defined on $\mathbb{R}$ by: $f(x) = a + (x+b)e^{-x}$ where a and b are two constant real numbers.



Therefore, based on all this information, we can assert that:

a) $a = -2$ and $b = -4$.

b) $a = -2$ and $b = 4$.

c) $a = 2$ and $b = 4$.

d) None of the three previous statements is correct.

23. In the set of complex numbers, the equation $z - \bar{z} + 2 - 4i = 0$ admits :

a) one unique solution. **b)** an infinity of solutions. **c)** two solutions. **d)** no solution.

24. Let f be the function defined on $\mathbb{R}$ by $f(x) = e^{(x-1)\ln(5)} - e^{(x-3)\ln(5)}$.

a) f converges to 0 as x tends to $+\infty$.

b) f diverges to $+\infty$, as x tends to $+\infty$.

c) For every $x \in \mathbb{R}$, we have $f(x) = 25$.

d) None of the three previous statements is correct.

25. Let (C) be the representing curve of the real function $f: x \longmapsto x - \sqrt{x^2 - x}$.

a) The straight line of equation $y = 2x$ is an oblique asymptote to (C) in the neighborhood of $-\infty$.

b) The straight line of equation $y = 2x - \frac{1}{2}$ is an oblique asymptote to (C) in the neighborhood of $-\infty$.

c) The curve (C) admits a vertical asymptote of equation $x = 1$.

d) None of the three previous statements is correct.

26. An unknown code consists of 8 characters. Each character can be a letter or a number. There are therefore 36 usable characters for each of the positions. A certain code cracking software can test about one hundred million codes per second.

a) The maximum time after which the software can discover the code is 3 hours.

b) The maximum time after which the software can discover the code is 8 hours.

c) The maximum time after which the software can discover the code is 3.4 hours.

d) None of the three previous statements is correct.

27. Consider the real function f defined on $\mathbb{R}$ by $f(x) = x^3 - 3x^2 + 3x - 5$.

What is the correct statement regarding the function f evaluated at point 1?

a) The function f admits a maximum when $x = 1$.

b) The function f admits a minimum when $x = 1$.

c) The curve of f admits an inflection point at $x = 1$.

d) None of the three previous statements is correct.

28. For the Boisjoli village festival, the mayor invited children from neighboring villages. The services of the town hall having managed the registrations, has counted 400 children at this party; they also indicate that 32% of the children present are children who live in the village of Boisjoli. During this party, eight children are chosen at random to form a team that will participate in a sporting challenge. It is assumed that the number of children is large enough for this situation to be likened to a draw with replacement. We call X the random variable taking as values the number of children in the team living in the village of Boisjoli.

The probability (*within 0.001*) that in the team there is at least one child living in the village of Boisjoli is:

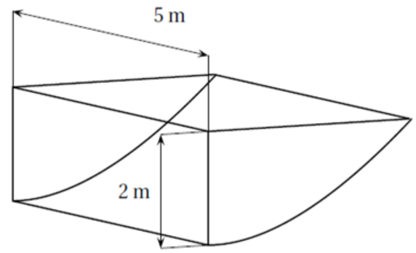
a) 0.125 **b)** 0.875 **c)** 0.954 **d)** 1

Exercise 2: (6 points)

An individual wants to have a water collector made. This water collector is a tank which must comply with the following specifications:

- it must be located two meters from his house;
- the maximum depth must be two meters;
- it must be five meters long;
- it must follow the natural slope of the ground.

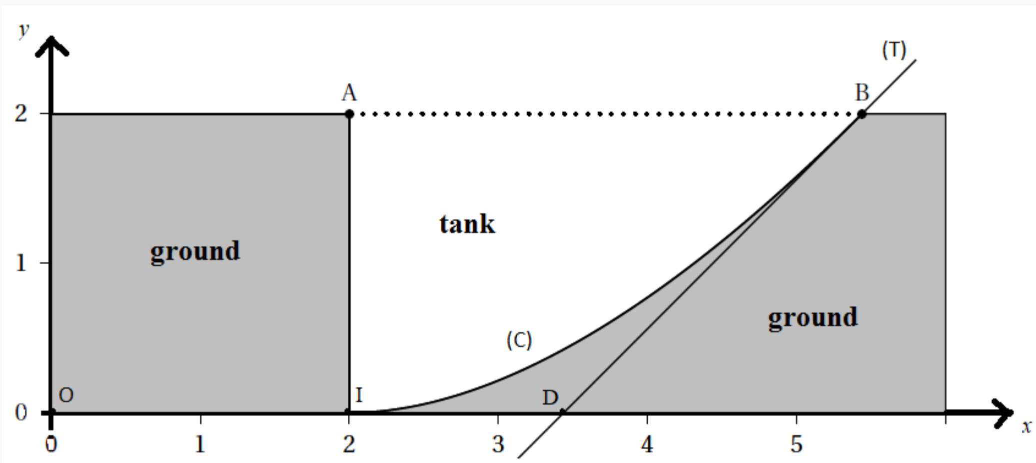
This tank is shown schematically on the right.



The curved part is modeled by curve (C) representing the function f defined on the interval $[2\,;2e]$ by:

$$f(x) = x\ln\left(\frac{x}{2}\right) - x + 2.$$

The curve (C) is represented below in an orthonormal system of **unit 1 meter** and it constitutes a profile view of the tank. Consider then the points $\mathrm{A}(2;2)$, $\mathrm{I}(2;0)$ and $\mathrm{B}(2e;2)$.



Part A

The purpose of this part is to calculate the volume of the tank.

1. Justify that the points B and I belong to the curve (C), and that the abscissa axis is tangent to (C) at point I.

2. Prove that the equation $f(x) = 1$ admits one unique solution x_0 in the interval $[2; 2e]$.

3. Show that $4.311 < x_0 < 4.312$. Deduce an approximated value of x_0 with 3 decimal places.

4. Study the concavity of the function f.

5. Let (T) be the tangent line to the curve (C) at point B, and let D be the point of intersection of line (T) with the abscissa axis.

a) Determine an equation of line (T) and deduce the coordinates of point D.

b) Let S be the area of the domain bounded by the curve (C) and the lines of equations $y = 2$ and $x = 2$.
Let V be the total volume of the tank. Therefore $V=5S$.
By noticing that S is bounded by the area of the triangle ABI and that of the trapezoid AIDB, deduce that $10e - 10 \leq V \leq 20e - 30$.

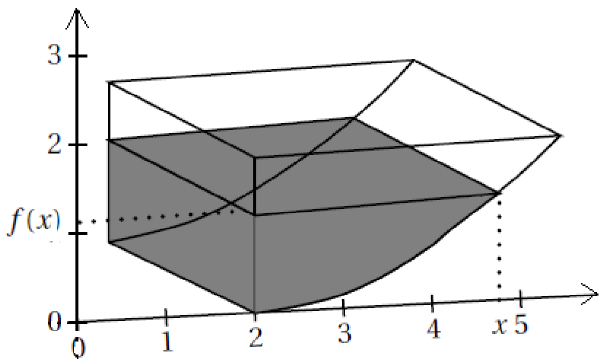
6. Let F be the function defined on the intervalle $[2; 2e]$ by : $F(x) = \frac{x^2}{2}\ln\left(\frac{x}{2}\right) - \frac{3x^2}{4} + 2x$.

a) Show that the function F is a primitive of f on $[2; 2e]$, meaning $F' = f$.

b) Let S' be the area of the domain bounded by the curve (C), the abscissa axis and the lines of equations $x = 2$ and $x = 2e$.
It can be shown in calculus that $S' = F(2e) - F(2)$.
Determine then the exact value of the area S and deduce an approximated value of V, in m^3, rounded to the nearest integer.

Part B

For any real x between 2 and $2e$, we denote $v(x)$ to be the volume of water, expressed in m^3, found in the tank when the height of water in the tank is equal to $f(x)$.



It can be shown that, for any real number $x \in [2; 2e]$, we have: $v(x) = 5\left[\frac{x^2}{2}\ln\left(\frac{x}{2}\right) - 2x\ln\left(\frac{x}{2}\right) - \frac{x^2}{4} + 2x - 3\right]$.

1. What volume of water, rounded to the nearest integer, is there in the tank when the height of water in the tank is 1 meter?

2. Let's remember that V is the total volume of the tank, f is the function defined at the beginning of the exercise and v is the function defined in part B.

a) Show that the function v is strictly increasing on the interval $[2; 2e]$.

b) Prove that there exists a unique value c in the interval $[2; 2e]$ such that $v(c) = \frac{V}{2}$.

c) So what does the value $f(c)$ represent?

Université Libanaise
Faculté de Génie

-CORRECTION-

Concours d'entrée

MATHÉMATIQUES

SUJET A

(Programme Libanais)

Date: *16/07/2022*
Durée: *3h*

Exercice 1:

Question	Réponse
1	c
2	c
3	b
4	d
5	c
6	b
7	c
8	b
9	c
10	b
11	b
12	b
13	a
14	c
15	b
16	c
17	b
18	b
19	c
20	c
21	d
22	b
23	d
24	b
25	b
26	b
27	c
28	c

Exercice 2:

Partie A:

1. ■ . $f(x_B) = f(2e) = 2e\ln\left(\frac{2e}{2}\right) - 2e + 2 = 2 = y_B$ donc $B \in (C)$.

. $f(x_I) = f(2) = 2\ln\left(\frac{2}{2}\right) - 2 + 2 = 0 = y_I$ donc $I \in (C)$.

■ $\forall x \in [2; 2e]$, $f'(x) = \ln\left(\frac{x}{2}\right) + x\left(\frac{1/2}{x/2}\right) - 1 = \ln\left(\frac{x}{2}\right)$ $\Rightarrow f'(2) = \ln\left(\frac{2}{2}\right) = 0$.

Donc la tangente à (C) en I est horizontale, d'où l'axe des abscisses est tangent à (C) en I.

2. $\forall x \in]2; 2e]$, on a: $x > 2 \Rightarrow \frac{x}{2} > 1 \Rightarrow \ln\left(\frac{x}{2}\right) > 0 \Rightarrow f'(x) > 0$. De plus $f'(2) = 0$.

Donc f est strictement croissante sur $[2; 2e]$.

Par ailleurs, f est continue sur $[2; 2e]$, et de plus, $\begin{cases} f(2) = 0 \\ f(2e) = 2 \end{cases}$.

Or $1 \in [0; 2]$, donc, d'après le théorème des valeurs intermédiaires, l'équation $f(x) = 1$ admet une unique solution $x_0 \in [2; 2e]$.

3. $f(4{,}311) = 0{,}99994 \ldots < 1$ et $f(4{,}312) = 1{,}000714 \ldots > 1$ $\Rightarrow \boxed{4{,}311 < x_0 < 4{,}312}$

On peut donc prendre $\boxed{x_0 \cong 4{,}311}$ comme valeur approchée de x_0 à 10^{-3} près.

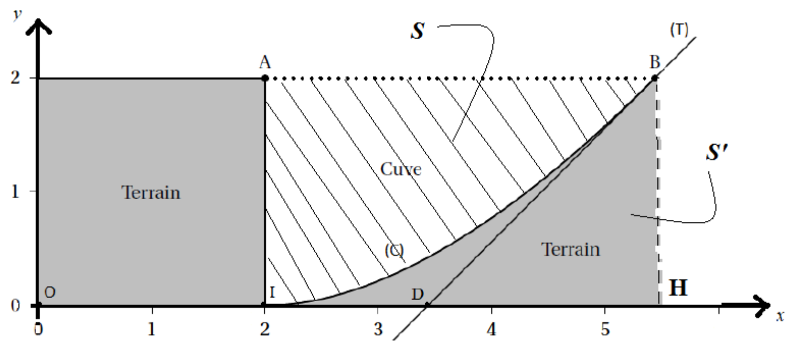
4. $\forall x \in [2; 2e]$, $f''(x) = \frac{1/2}{x/2} = \frac{1}{x} > 0$ donc f est convexe sur $[2; 2e]$.

5. **a)** ■ (T): $y = f'(2e)(x - 2e) + f(2e)$

$\begin{cases} f(2e) = 2 \\ f'(2e) = \ln(e) = 1 \end{cases} \Rightarrow (T)\colon\ y = x - 2e + 2$

■ Pour $y = 0$ on a: $x = 2e - 2$. Donc $\boxed{D(2e - 2; 0)}$.

b)



· Aire du triangle ABI: $\mathcal{A}_{\text{ABI}} = \frac{\text{AI}\times\text{AB}}{2} = \frac{2\times(2e-2)}{2} = 2e - 2$ m²

· Aire du trapèze AIDB: $\mathcal{A}_{\text{AIDB}} = \frac{\text{AI}\times(\text{AB}+\text{ID})}{2} = \frac{2\times(2e-2+2e-4)}{2} = 4e - 6$ m²

· Or $\mathcal{A}_{\text{ABI}} \leq S \leq \mathcal{A}_{\text{AIDB}}$ donc $2e - 2 \leq S \leq 4e - 6$ $\Rightarrow 5(2e - 2) \leq 5S \leq 5(4e - 6)$.

D'où $\boxed{10e - 10 \leq V \leq 20e - 30}$. *cqfd*

6. a) $\forall x \in [2;2e]$, $F'(x) = x\ln\left(\frac{x}{2}\right) + \frac{x^2}{2} \times \frac{1/2}{x/2} - \frac{3x}{2} + 2$

$= x\ln\left(\frac{x}{2}\right) + \frac{x}{2} - \frac{3x}{2} + 2$

$= x\ln\left(\frac{x}{2}\right) - x + 2$

$= f(x)$

Donc F est bien une primitive de f sur $[2;2e]$.

b) · $S' = F(2e) - F(2) = \left(2e^2\ln(e) - \frac{3}{4}(4e^2) + 4e\right) - \left(2\ln(1) - \frac{3}{4}(4) + 4\right)$

$= 2e^2 - 3e^2 + 4e + 3 - 4$

$\boxed{= -e^2 + 4e - 1}$

· Soit H le projeté orthogonal de B sur (Ox).

On a alors $S + S' =$ aire du rectangle AIHB $=$ AI $\times$ AB $= 2(2e-2) = 4e - 4$.

D'où $S = 4e - 4 - S'$

$= 4e - 4 - (-e^2 + 4e - 1)$

$= e^2 - 3 \ m^2$

On en déduit que $V = 5(e^2 - 3) = \boxed{5e^2 - 15}$; donc on peut affirmer que $\boxed{V \cong 22\ m^3}$

Partie B:

1. D'après la partie A, on sait que $f(x) = 1 \iff x = x_0 \cong 4{,}311$.

Pour $x = x_0$, le volume de la cuve vaut: $v(x_0) = v(4{,}311) \cong \boxed{7\ m^3}$

2. a) $\forall x \in [2;2e]$, $v'(x) = 5\left[x\ln\left(\frac{x}{2}\right) + \frac{x^2}{2} \times \frac{\frac{1}{2}}{\frac{x}{2}} - 2\ln\left(\frac{x}{2}\right) - 2x \times \frac{\frac{1}{2}}{\frac{x}{2}} - \frac{x}{2} + 2\right]$

$= 5(x-2)\ln\left(\frac{x}{2}\right)$.

D'autre part, $2 < x \le 2e \implies x - 2 > 0 \implies \frac{x}{2} > 1 \implies \ln\left(\frac{x}{2}\right) > 0 \implies v'(x) > 0$; De plus $v'(2) = 0$.

Donc la fonction v est strictement croissante sur l'intervalle $[2;2e]$.

b) Sur $[2;2e]$, v est continue et strictement croissante. De plus $\begin{cases} v(2) = -1 + 4 - 3 = 0 \\ v(2e) = 5(e^2 - 3) = V \end{cases}$.

Or $\frac{V}{2} \in [0;V]$, donc l'équation $v(x) = \frac{V}{2}$ admet une solution unique $c \in [2;2e]$.

c) $f(c)$ représente la hauteur d'eau correspondant au remplissage à moitié de la cuve.