UNIVERSITE LIBANAISE

FACULTE DE GENIE

الجامعة اللبنانية
كلية الهندسة

Entrance Exam 2013 - 2014 — *Mathematics* — *Duration : 3 hours*

The distribution of grades is over 25 — *July 13 , 2013*

I- (2.5 pts) The space is referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.

Consider in the plane (P) of equation $2x+y-2z+3=0$, the circle (C) of center $A(1;-3;1)$ and radius $\sqrt{3}$; and in the plane (Q) of equation $x-y-z-3=0$, the circle (γ) of center $B(2;-1;0)$ and radius 3.

1- Write a system of parametric equations of each of the axis (d) of (C) and the axis (δ) of (γ) .

2- Determine the point of intersection I of (d) and (δ) .

3- Prove that I is the center of a sphere (S) containing the circles (C) and (γ). Calculate the volume of (S) .

II- (3.5 pts) Consider the equation (E) : $(\cos^2\alpha)z^2+(\sin 2\alpha)z+1+\sin^2\alpha=0$ where $0\le\alpha<\dfrac{\pi}{2}$.

Let M' and M'' be the images , in the complex plane , of the solutions z' and z'' of (E) .

1- Calculate z' and z'' in terms of α and prove that , as α varies , $z'^2+z''^2$ remains constant .

2- Calculate $M'M''$ in terms of α and determine α so that $M'M''$ is minimum .

3- Prove that , as α varies , M' and M'' vary on a hyperbola (H) of center O , for which the asymptotes , a focus and the associated directrix are to be determined . Draw (H) .

III- (3.5 pts) Consider the sequences (U_n) , (V_n) and (W_n) defined for all natural numbers $n\ge 1$ by

$$U_n=\frac{1^3}{n^4}+\frac{2^3}{n^4}+\cdots\cdots+\frac{n^3}{n^4}\ ;\ V_n=\frac{1}{n^2}+\frac{2}{n^2}+\cdots\cdots+\frac{n}{n^2}\ \text{and}\ W_n=\sin\frac{1}{n^2}+\sin\frac{2}{n^2}+\cdots\cdots+\sin\frac{n}{n^2}.$$

1- Prove that (U_n) has 1 as an upper bound and that (V_n) converges to $\dfrac{1}{2}$.

2- a) Using the inequality (1) : $x-\dfrac{x^3}{6}\le\sin x\le x$ which is true for all x in $[0;+\infty[$, prove that :

For all $n\ge 1$ and for all natural numbers k , $\dfrac{k}{n^2}-\dfrac{1}{6n^2}\times\dfrac{k^3}{n^4}\le\sin\dfrac{k}{n^2}\le\dfrac{k}{n^2}$.

b) Prove that , for all $n\ge 1$, $V_n-\dfrac{1}{6n^2}\times U_n\le W_n\le V_n$ and deduce that $V_n-\dfrac{1}{6n^2}\le W_n\le V_n$.

c) Prove that (W_n) is convergent and determine its limit .

IV- (3.5 pts) Consider an urn containing 10 balls of which n balls are green , m balls are red and the others are white such that $n\ge 2$; $m\ge 2$ and $n+m\le 8$.

A player pays $5\ \$$ and draws two balls at random from the urn; he gains $15\ \$$ for each green ball drawn , $5\ \$$ for each red ball drawn and loses $5\ \$$ for each white ball drawn .

Let X be the random variable that represents the total algebraic gain of the player after the game .

1- a) Determine the values of X .

b) Calculate $p(X=25)$ and $p(X=15)$ in terms of n and m .

c) Knowing that $p(X=25)=\dfrac{1}{15}$ and $p(X=15)=\dfrac{2}{15}$, determine n and m .

2- Suppose in this part that the urn contains 3 green balls , 2 red balls and 5 white balls .

a) Determine the probability distribution of X and calculate its expected value .

b) Calculate the probability that the player has drawn 2 balls of same color knowing that his total algebraic gain was positive .

V- (5 pts) Given in an oriented plane , a circle (C) of center A and radius 3 and a circle (C') of center B and radius 1 such that $AB=6$.

1- Let S be the similitude of angle $\dfrac{\pi}{3}$ that transforms (C) into (C') .

a) Determine the ratio of S and justify that its center I is such that $IA=3IB$.

b) Prove that $IA=\dfrac{18}{\sqrt{7}}$ and $IB=\dfrac{6}{\sqrt{7}}$. Construct I .

2- Let r be the rotation of center A and angle $\dfrac{2\pi}{3}$ and h the dilation of center A and ratio $\dfrac{2}{3}$.

a) Construct the points D and E such that $D=r(B)$ and $E=h(B)$.

b) Calculate $\dfrac{BE}{AD}$ and $(\overrightarrow{AD}\ ;\ \overrightarrow{BE})$. Deduce $S(D)$.

c) Prove that I belongs to the circle circumscribed about the triangle ADE .

In what follows , refer the plane to the direct orthonormal system $(A\ ;\ \vec{u}\ ,\ \vec{v})$ ***such that*** $\vec{u}=\dfrac{1}{6}\overrightarrow{AB}$ **.**

3- Determine the complex relation of the similitude S . Deduce the affix of I .

4- a) Determine the complex relation of each of the rotation r and the dilation h .

b) Determine the affix of each of the points D and E and verify that $S(D)=E$.

VI- (7 pts) Consider the function f defined on the interval $]0\ ;+\infty[$ by $f(x)=\ell n^2 x-\ell n x$.

Let (C) be the representative curve of f in an orthonormal system $(O\ ;\ \vec{i}\ ,\ \vec{j})$.

1- Determine the points of intersection A and B , ($x_A < x_B$) , of (C) and the axis of abscissas .

2- a) Set up the table of variations of f and determine the point S corresponding to the minimum of f .

b) Prove that the restriction of f to the interval $]0\,;1]$ has an inverse function f^{-1} to be determined .

3- a) Study the concavity of (C) and determine its point of inflection I .

b) Verify that the abscissas of the points A , B , S and I are , in a certain order , 4 consecutive terms of an increasing geometric sequence whose common ratio is to be determined .

4- Draw (C) . (***Unit : 2 cm***)

5- a) Determine , in terms of α , an equation of the tangent (d) to (C) at the point M of abscissa α .

b) Determine the ordinate β of the point of intersection of (d) with the axis of ordinates .

c) Prove that , as α traces $]0\,;+\infty[$, β has a minimum β_0 . Determine β_0 and the corresponding position of M .

6- a) Prove that , for all $m > \beta_0$, there exists two points M_1 and M_2 on (C) where the tangent to (C) cuts the axis of ordinates at the point with ordinate m .

b) Prove that the abscissas α_1 and α_2 of M_1 and M_2 are such that $\alpha_1\alpha_2 = e^3$.

c) Determine the point E of (C) such that the tangents to (C) at E and B intersect on the axis of ordinates .

7- Consider the sequence (I_n) defined on IN by $I_n = \int_1^e (\ell n\, x)^n\, dx$.

a) Using integration by parts , prove that , for all $n \geq 1$, $I_n = e - n\,I_{n-1}$.

b) Calculate the area of the domain bounded by (C) and the axis of abscissas in cm^2 .

UNIVERSITE LIBANAISE

FACULTE DE GENIE

الجامعة اللبنانية
كلية الهندسة

Entrance Exam 2013 - 2014 | ***Solution of Mathematics*** | ***Duration : 3 hours***

The distribution of grades is over 25 | ***July 13 , 2013***

EXERCISE 1

1- The axis (d) of (C) is the perpendicular to (P) at A ; $\overrightarrow{u}(2\,;1\,;-2)$ is a direction vector of (d).
A system of parametric equations of (d) is $(x=2t+1\ ;\ y=t-3\ ;\ z=-2t+1\ ;\ t\in IR)$

The axis (δ) of (γ) is the perpendicular to (Q) at B ; $\overrightarrow{v}(1\,;-1\,;-1)$ is a direction vector of (δ).
A system of parametric equations of (δ) is $(x=m+2\ ;\ y=-m-1\ ;\ z=-m\ ;\ m\in IR)$.

2- The system $(2t+1=m+2\ ;\ t-3=-m-1\ ;\ -2t+1=-m)$ has a unique solution $m=t=1$.
Therefore , (d) and (δ) intersect at the point $I(3\,;-2\,;-1)$.

3- I belongs to (d) then I is equidistant from all points of (C); For any point M of (C) , the triangle IAM is right at A such that $IA=\sqrt{4+1+4}=3$ and $AM=r=\sqrt{3}$ then $IM=\sqrt{IA^2+AM^2}=\sqrt{12}=2\sqrt{3}$.
I belongs to (δ) then I is equidistant from all points of (γ); For any point M of (γ) , the triangle IBM is right at B such that $IB=\sqrt{1+1+1}=\sqrt{3}$ and $BM=r'=3$ then $IM=\sqrt{IB^2+BM^2}=\sqrt{12}=2\sqrt{3}$.
Therefore I is equidistant from all points of $(C)\cup(\gamma)$. Hence , I is the center of a sphere (S) of radius $R=2\sqrt{3}$ containing the circles (C) and (γ) .

EXERCISE 2

1- (E) : $(\cos^2\alpha)z^2+2(\sin\alpha\cos\alpha)z+1+\sin^2\alpha=0$; for all $[0\,;\frac{\pi}{2}[$, the equation (E) is quadratic .

$\Delta'=\sin^2\alpha\cos^2\alpha-\cos^2\alpha-\sin^2\alpha\cos^2\alpha=-\cos^2\alpha=i^2\cos^2\alpha$.

The solutions of (E) are $z'=\dfrac{-\sin\alpha\cos\alpha+i\cos\alpha}{\cos^2\alpha}=-\tan\alpha+\dfrac{1}{\cos\alpha}i$ and $z''=-\tan\alpha-\dfrac{1}{\cos\alpha}i$.

$$z'^2+z''^2=\left(-\tan\alpha+\frac{1}{\cos\alpha}i\right)^2+\left(-\tan\alpha-\frac{1}{\cos\alpha}i\right)^2=2\left(\tan^2\alpha-\frac{1}{\cos^2\alpha}\right)=2(-1)=-2.$$

OR $$z'^2+z''^2=(z'+z'')^2-2z'z''=(-2\tan\alpha)^2-2\left(\frac{1}{\cos^2\alpha}-\tan^2\alpha\right)=-2(1)=-2\ .$$

2- $M'M''=|z'-z''|=\left|\dfrac{2}{\cos\alpha}i\right|=\left|\dfrac{2}{\cos\alpha}\right|=\dfrac{2}{\cos\alpha}$ since $0\le\alpha<\dfrac{\pi}{2}$ then $\cos\alpha>0$.
$M'M''$ is minimum is equivalent to $\cos\alpha$ is maximum where $0<\cos\alpha\le1$; therefore $M'M''$ is minimum when $\cos\alpha=1$; that is when $\alpha=0$

3- $M'(-\tan\alpha\ ;\ \dfrac{1}{\cos\alpha})$ and $M''(-\tan\alpha\ ;\ \dfrac{-1}{\cos\alpha})$ are the images of z' and z''

The coordinates x and y of each of M' and M'' are such that $x^2-y^2=\tan^2\alpha-\dfrac{1}{\cos^2\alpha}=-1$.

Therefore , as α varies , M' and M'' vary on the hyperbola (H) of equation $y^2-x^2=1$.

The center of (H) is the origin O , the asymptotes are the straight lines of equations $y=x$ and $y=-x$.

The focal axis of (H) is the axis of ordinates .

$a=b=1$ then $c=\sqrt{2}$; therefore $F(0\ \sqrt{2})$ is a focus of (H) and the straight line (d) of equation $y=\dfrac{a^2}{c}=\dfrac{\sqrt{2}}{2}$ is the associated directrix .

Drawing (H) .

EXERCISE 3

1- $U_n=\dfrac{1}{n^4}+\dfrac{2^3}{n^4}+\dfrac{3^3}{n^4}+\cdots\cdots\cdots+\dfrac{n^3}{n^4}$ then $U_n\le\underbrace{\dfrac{n^3}{n^4}+\dfrac{n^3}{n^4}+\dfrac{n^3}{n^4}+\cdots\cdots\cdots+\dfrac{n^3}{n^4}}_{n\ times}=n\left(\dfrac{n^3}{n^4}\right)=1$

$V_n=\dfrac{1+2+3+\cdots\cdots+n}{n^2}=\dfrac{n(n+1)}{2n^2}$ then $\lim\limits_{n\to+\infty}V_n=\lim\limits_{n\to+\infty}\dfrac{n^2}{2n^2}=\dfrac{1}{2}$ and (V_n) converges to $\dfrac{1}{2}$.

2- The sequence (W_n) is defined for $n\ge1$ by $W_n=\sin\dfrac{1}{n^2}+\sin\dfrac{2}{n^2}+\sin\dfrac{3}{n^2}+\cdots\cdots\cdots+\sin\dfrac{n}{n^2}$.

a) By applying (1) to $\dfrac{1}{n^2}$ we get $\dfrac{1}{n^2}-\dfrac{1^3}{6n^6}\le\sin\dfrac{1}{n^2}\le\dfrac{1}{n^2}$; that is $\dfrac{1}{n^2}-\dfrac{1}{6n^2}\times\dfrac{1^3}{n^4}\le\sin\dfrac{1}{n^2}\le\dfrac{1}{n^2}$.

b) By applying (1) to $\dfrac{k}{n^2}$ for $k\in\{1\,;\,2\,;\,3\cdots\cdots\cdots;\,n\}$ and adding the n inequalities we get

$V_n-\dfrac{1}{6n^2}\times U_n\le W_n\le V_n$.

For all $n\ge1$, $U_n\le1$ then $V_n-\dfrac{1}{6n^2}\le W_n\le V_n$.

c) (V_n) converges to $\dfrac{1}{2}$ and $\lim\limits_{n\to+\infty}\dfrac{1}{6n^2}=0$ then (W_n) is convergent and its limit is equal to $\dfrac{1}{2}$.

EXERCISE 4

1- The random variable X represents the total algebraic gain of the player after the game .

a) ▪ If the player draws 2 green balls then , $X = 15+15-5=25$.
▪ If the player draws a green ball and a red one then , $X = 15+5-5=15$.
▪ If the player draws a green ball and a white one then , $X = 15-5-5=5$.
▪ If the player draws 2 red balls then , $X = 5+5-5=5$.
▪ If the player draws a red ball and a white one then , $X = 5-5-5=-5$.
▪ If the player draws 2 white balls then $X = -5-5-5=-15$.
Hence , the set of values of X is $\{-15\,;\,-5\,;\,5\,;\,15\,;\,25\}$.

b) When 2 balls are randomly drawn from the urn that contains 10 balls , the sample space is equiprobable and consists of ${}_{10}C_2$ possible outcomes .
▪ $(X=25)$ represents the event " the player draws 2 green balls " ; therefore
$p(X=25)=\dfrac{{}_nC_2}{{}_{10}C_2}=\dfrac{n(n-1)}{90}$.
▪ $(X=15)$ represents the event " the player draws a green ball and a red one " ; therefore
$p(X=15)=\dfrac{n\times m}{{}_{10}C_2}=\dfrac{n\times m}{45}$.

c) $p(X=25)=\dfrac{1}{15}$ is equivalent to $\dfrac{n(n-1)}{90}=\dfrac{1}{15}$; $n(n-1)=6$ therefore $n=3$.
$p(X=15)=\dfrac{2}{15}$ is equivalent to $\dfrac{n\times m}{45}=\dfrac{2}{15}$; $mn=6$ where $n=3$; therefore $m=2$.

2- Suppose in this part that the urn contains 3 green balls , 2 red balls and 5 white balls .
a) ▪ $(X=-15)$ is the event " the player draws 2 white balls " ; therefore $p(X=-15)=\dfrac{{}_5C_2}{{}_{10}C_2}=\dfrac{2}{9}$.
▪ $(X=-5)$ is the event " the player draws 1 red ball and 1 white one " ; $p(X=-5)=\dfrac{2\times 5}{{}_{10}C_2}=\dfrac{2}{9}$.

- $(X=5)$ is the event " the player draws 1 green ball and 1 white one or 2 red balls " ;
  $p(X=5)=\frac{3\times 5}{{}_{10}C_2}+\frac{{}_2C_2}{{}_{10}C_2}=\frac{16}{45}$.
- $p(X=15)=\frac{2}{15}$ and $p(X=25)=\frac{1}{15}$

The expected gain of the player is $\overline{X}=-15\times\frac{2}{9}-5\times\frac{2}{9}+5\times\frac{16}{45}+15\times\frac{2}{15}+25\times\frac{1}{15}=1\,\$$.

b) Let A : " the player draws 2 balls of same color " and B : " the algebraic gain is positive " .
The required probability is $p(A/B)=\frac{p(A\cap B)}{p(B)}$ where
$A\cap B$: " the player draws 2 green balls or 2 red balls " ;
$p(A\cap B)=p(X=25)+\frac{{}_2C_2}{{}_{10}C_2}=\frac{1}{15}+\frac{1}{45}=\frac{4}{45}$ and
$p(B)=p(X=5)+p(X=15)+p(X=25)=\frac{25}{45}=\frac{5}{9}$.
Therefore $p(A/B)=\frac{p(A\cap B)}{p(B)}=\frac{4}{25}$.

EXERCISE 5

1- S is the similitude of center I angle $\frac{\pi}{3}$ that transforms (C) into (C').

a) - The ratio of S is $k=\frac{radius\ of\ (C')}{radius\ of\ (C)}=\frac{1}{3}$.

- The similitude transforms the center A of (C) into the center B of (C') ; therefore $IB=\frac{1}{3}IA$; that is $IA=3IB$.

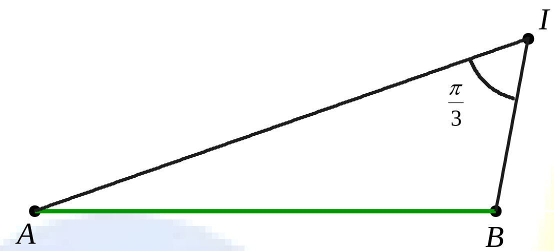
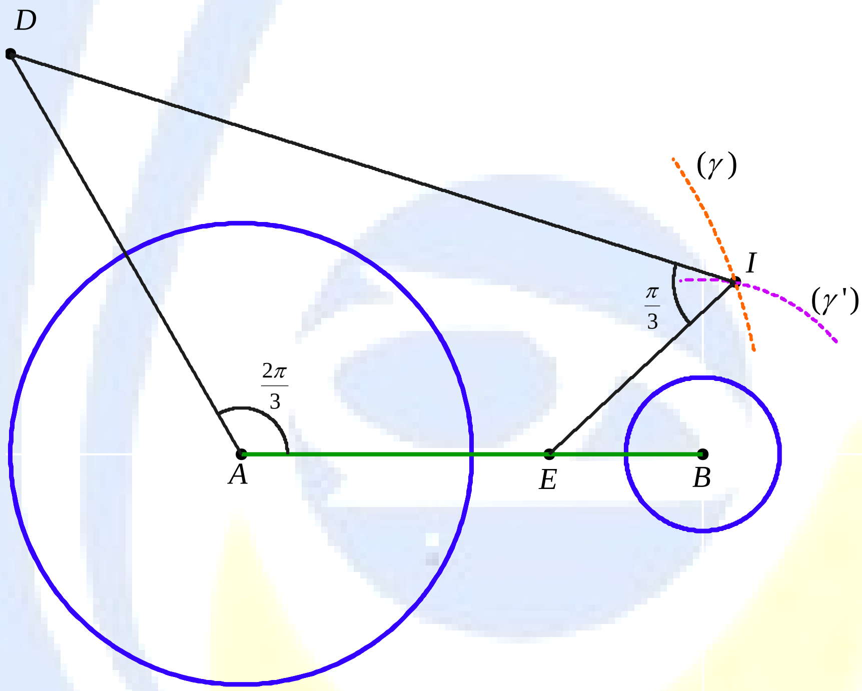


Figure 10

b) $S(A)=B$; then $(\overrightarrow{IA}\,;\overrightarrow{IB})=\frac{\pi}{3}\ (2\pi)$.

- In triangle IAB we can write
  $AB^2=IA^2+IB^2-2\,IA\times IB\times\cos\frac{\pi}{3}$; that is
  $36=9IB^2+IB^2-3IB^2$; $IB^2=\frac{36}{7}$.
  Therefore $IB=\frac{6}{\sqrt{7}}$ and $IA=\frac{18}{\sqrt{7}}$.
- The points A and B being given , the point I belongs to the circle (γ) of center A and radius $\frac{18}{\sqrt{7}}$ and the circle (γ') of center B and radius $\frac{6}{\sqrt{7}}$.

The circles (γ) and (γ') intersect at two points ; I is the point such that $(\overrightarrow{IA}\ ;\ \overrightarrow{IB}) = +\dfrac{\pi}{3}\ \ (2\pi)$.



2- Consider the rotation $r = r(A\ ,\ \dfrac{2\pi}{3})$ and the dilation $h = h(A\ ,\ \dfrac{2}{3})$.

a) ▪ $D = r(B)$; therefore D is the point such that $AD = AB = 6$ and $(\overrightarrow{AB}\ ;\ \overrightarrow{AD}) = \dfrac{2\pi}{3}\ \ (2\pi)$.

▪ $E = h(B)$; therefore E is the point such that $\overrightarrow{AE} = \dfrac{2}{3}\overrightarrow{AB}$; therefore E is the point of $[AB]$ such that $AE = 4$ and $BE = 2$.

b) ▪ $\dfrac{BE}{AD} = \dfrac{2}{6} = \dfrac{1}{3}$.

▪ $(\overrightarrow{AD}\ ;\ \overrightarrow{BE}) = (\overrightarrow{AD}\ ;\ \overrightarrow{BA}) = -(\overrightarrow{BA}\ ;\ \overrightarrow{AD}) = -(\overrightarrow{AB}\ ;\ \overrightarrow{AD}) + \pi = -\dfrac{2\pi}{3} + \pi = \dfrac{\pi}{3}\ \ (2\pi)$.

▪ The above relations with $S(A) = B$ show that $S(D) = E$.

c) $S(D)=E$ gives $(\overrightarrow{ID}\ ;\ \overrightarrow{IE})=\frac{\pi}{3}\ \ (2\pi)$.

The quadrilateral $AEID$ is cyclic for having two supplementary opposite angles $E\hat{I}D$ and $E\hat{A}D$; therefore I belongs to the circle circumscribed about the triangle ADE .

The plane is referred to the direct orthonormal system $(A\ ;\ \overrightarrow{u}\ ,\ \overrightarrow{v})$ ***such that*** $\overrightarrow{u}=\frac{1}{6}\overrightarrow{AB}$ **.**

3- In this system we have $A(0\ ;\ 0)$, $B(6\ ;\ 0)$

The complex relation of the similitude $S(I\ ;\ \frac{1}{3}\ ;\ \frac{\pi}{3})$ is of the form $z'=az+b$ where

- $a=\frac{1}{3}e^{i\frac{\pi}{3}}=\frac{1}{3}(\frac{1}{2}+\frac{\sqrt{3}}{2}i)=\frac{1}{6}+\frac{\sqrt{3}}{6}i$.
- $B=S(A)$; that is $z_B=az_A+b$; $6=b$.

Therefore the complex relation of S is $z'=(\frac{1}{6}+\frac{\sqrt{3}}{6}i)z+6$.

The affix of the center I of S is $z_I=\frac{b}{1-a}=\frac{6}{\frac{5}{6}-\frac{\sqrt{3}}{6}i}=\frac{36}{5-\sqrt{3}i}=\frac{36(5-\sqrt{3}i)}{28}=\frac{45}{7}-\frac{9\sqrt{3}}{7}i$.

4- a) The complex relation of the rotation $r(A\ ;\ \frac{2\pi}{3})$ is of the form $z'=az+b$ where

- $a=e^{i\frac{2\pi}{3}}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$.
- $A=r(A)$; that is $b=0$.

Therefore the complex relation of r is $z'=(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)z$.

The complex relation of the dilation $h(A\ ;\ \frac{2}{3})$ is $z'=\frac{2}{3}z$.

b)
- $D=r(B)$; therefore $z_D=(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)z_B=6(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)=-3+3\sqrt{3}i$; $D(-3\ ;\ +3\sqrt{3})$.
- $E=h(B)$; therefore $z_E=\frac{2}{3}z_B=4$; $E(4\ ;\ 0)$.
- $(\frac{1}{6}+\frac{\sqrt{3}}{6}i)z_D+6=(\frac{1}{6}+\frac{\sqrt{3}}{6}i)(-3+3\sqrt{3}i)+6=-\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{\sqrt{3}}{2}i-\frac{3}{2}+6=4=z_E$; therefore $S(D)=E$.

EXERCISE 6

The function f is defined on the interval $]0\,;+\infty[$ by $f(x)=\ell n^2 x-\ell n\, x$.

1- The abscissas of the points of intersection of (C) and the axis of abscissas are the solutions of the equation $f(x)=0$ which is equivalent to $\ell n^2 x-\ell n\, x=0$; $\ell n\, x=0$ or $\ell n\, x=1$ then $x=1$ or $x=e$.
The points of intersection of (C) and $x'x$ are $A(1\,;0)$ and $B(e\,;0)$.

2- a) $\lim\limits_{x\to0^+}\ell n\, x=-\infty$ then $\lim\limits_{x\to0^+} f(x)=\lim\limits_{x\to0^+}(\ell n^2 x-\ell n\, x)=+\infty$.

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\ell n\, x\,(\ell n\, x-1)=+\infty$.

$f'(x)=\dfrac{2\ell n\, x-1}{x}$.

Table of variations of f

x	0		$\sqrt{e}$		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	↘	$-\frac{1}{4}$	↗	$+\infty$

Figure 18

The point of (C) corresponding to the minimum of f is $S(\sqrt{e}\,;-\dfrac{1}{4})$.

b) The restriction of f to the interval $]0\,;1]$ is continuous and strictly decreasing then , it has an inverse function f^{-1} defined on $f(]0\,;1])=[0\,;+\infty[$.

For all x in $[0\,;+\infty[$, $y=f^{-1}(x)$ is equivalent to $x=f(y)=\ell n^2 y-\ell n\, y$; that is $\ell n^2 y-\ell n\, y-x=0$ where $y\in]0\,;1]$ then $\ell n\, y\in]-\infty\,;0]$; therefore $\ell n\, y=\dfrac{1-\sqrt{1+4x}}{2}$ and $y=\exp\left(\dfrac{1-\sqrt{1+4x}}{2}\right)$.

Finally , f^{-1} is defined on $[0\,;+\infty[$ by $f^{-1}(x)=\exp\left(\dfrac{1-\sqrt{1+4x}}{2}\right)$.

3- a) $f''(x)=\dfrac{3-2\ell n\, x}{x^2}$.

Table of concavity of (C)

x	0		$e\sqrt{e}$		$+\infty$
$f''(x)$		$+$	0	$-$	
(C) concaves		*upwards*		*downwards*	

Figure 19

The concavity of (C) changes at the point $I(e\sqrt{e}\,;\dfrac{3}{4})$
Which is the point of inflection of (C).

b) The abscissas of the points A , S , B and I are respectively 1 , $\sqrt{e}$, e and $e\sqrt{e}$; these numbers are , in this order , 4 consecutive terms of an increasing geometric sequence of common ratio $\sqrt{e}$.

4- $\lim\limits_{x\to0^+} f(x)=+\infty$ then , the axis of ordinates is asymptote to (C) .

For all n in IN, $\lim\limits_{x\to+\infty}\dfrac{\ell n^n x}{x}=0$ then, $\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to+\infty}\left(\dfrac{\ell n^2 x}{x}-\dfrac{\ell n\, x}{x}\right)=0$; therefore (C) has at $+\infty$ an asymptotic direction parallel to the axis of abscissas .

Drawing (C) .

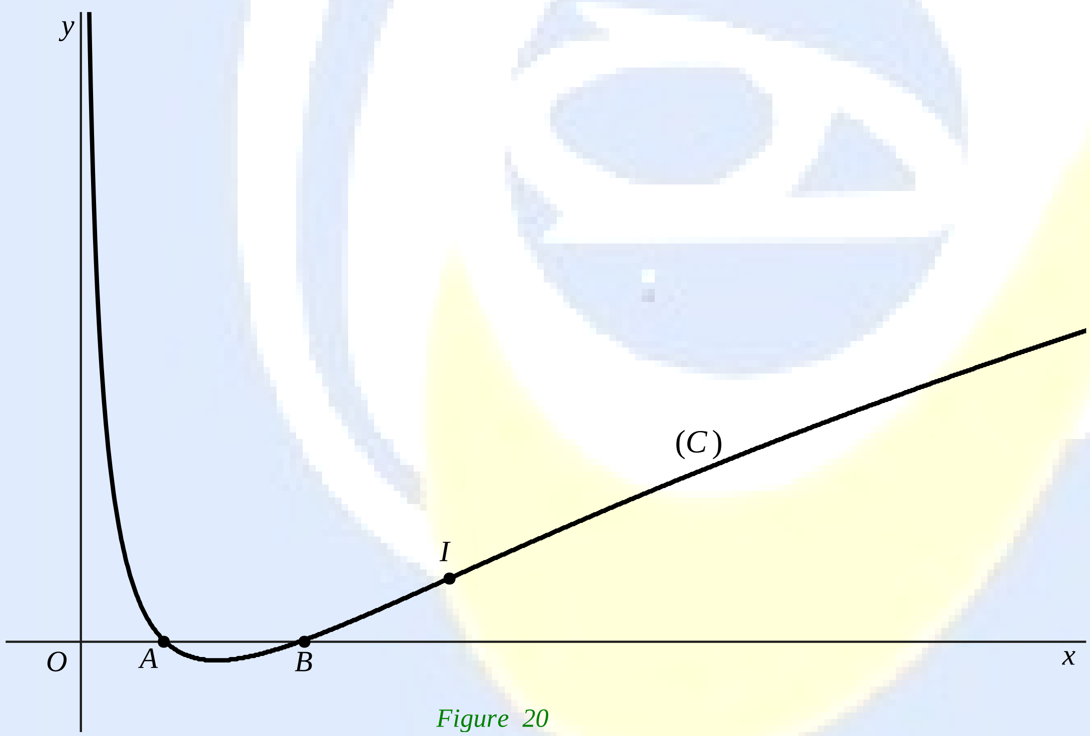


Figure 20

5- a) An equation of the tangent (d) to (C) at the point M of abscissa α is $y=f'(\alpha)(x-\alpha)+f(\alpha)$;

(d) : $y=\dfrac{2\ell n\alpha-1}{\alpha}(x-\alpha)+\ell n^2\alpha-\ell n\alpha$.

b) (d) cuts $y'y$ at the point of ordinate $\beta=\ell n^2\alpha-3\ell n\alpha+1$.

c) $\beta = \ell n^2\alpha - 3\ell n\alpha + 1 = \left(\ell n\alpha - \frac{3}{2}\right)^2 - \frac{5}{4}$ then , as α traces $]0\,;+\infty[$, β traces $[-\frac{5}{4}\,;+\infty[$ and takes its minimum value $\beta_0 = -\frac{5}{4}$ when $\ell n\alpha = \frac{3}{2}$; $\alpha = e\sqrt{e}$; that is $M = I$.

6- a) $\beta = m$ is equivalent to $\left(\ell n\alpha - \frac{3}{2}\right)^2 = m + \frac{5}{4}$.

For all $m > \beta_0$, the equation $\beta = m$ is equivalent to $\ell n\alpha - \frac{3}{2} = \sqrt{m+\frac{5}{4}}$ or $\ell n\alpha - \frac{3}{2} = -\sqrt{m+\frac{5}{4}}$;

then , there exists two points M_1 and M_2 on (C) with abscissas α_1 and α_2 such that $\ell n\alpha_1 = \frac{3}{2} + \sqrt{m+\frac{5}{4}}$ and $\ell n\alpha_2 = \frac{3}{2} - \sqrt{m+\frac{5}{4}}$ where the tangent to (C) cuts the axis of ordinates at the point with ordinate m .

b) $\ell n\alpha_1 + \ell n\alpha_2 = 3$ then $\ell n(\alpha_1\alpha_2) = 3$; that is $\alpha_1\alpha_2 = e^3$.

OR a) $\beta = m$ is equivalent to $\ell n^2\alpha - 3\ell n\alpha + 1 - m = 0$; $(\ell n\alpha)^2 - 3\ell n\alpha + 1 - m = 0$.

For the quadratic equation in $\ell n\alpha$: $(\ell n\alpha)^2 - 3\ell n\alpha + 1 - m = 0$, $\Delta = 4m + 5$ then ,

For all $m > \beta = -\frac{5}{4}$, this equation has two solutions in $\ell n\alpha$ and , since $\ell n\alpha$ can take any real value , therefore there exists two values of α for which $\beta = m$; hence there exists two points M_1 and M_2 on (C) where the tangent to (C) cuts the axis of ordinates at the point with ordinate m .

b) $\ell n\alpha_1$ and $\ell n\alpha_2$ are the solutions of the quadratic equation in $\ell n\alpha$: $(\ell n\alpha)^2 - 3\ell n\alpha + 1 - m = 0$; therefore $\ell n\alpha_1 + \ell n\alpha_2 = 3$ then $\ell n(\alpha_1 \times \alpha_2) = 3$; that is $\alpha_1\alpha_2 = e^3$.

c) The tangents to (C) at E and B intersect on the axis of ordinates if and only if the abscissa of E is such that $x_B \times x_E = e^3$ where $x_B = e$ then $x_E = e^2$; $E(e^2\,;\,2)$

7- a) Let $u(x) = (\ell n x)^n$ and $v'(x) = 1$ then $u'(x) = n\frac{(\ell n x)^{n-1}}{x}$ and $v(x) = x$; therefore

$$I_n = \int_1^e (\ell n x)^n dx = \left[x(\ell n x)^n\right]_1^e - n\int_1^e (\ell n x)^{n-1} dx = e - nI_{n-1} .$$

b) For all x in $[1\,;e]$, $f(x) \leq 0$ then , the required area S is such that $S = -\int_1^e f(x)\,dx$ *units of area* .

$$\int_1^e f(x)\,dx = \int_1^e (\ell n^2 x - \ell n\, x)dx = I_2 - I_1 \ .$$

$I_0 = \int_1^e dx = \left[\,x\,\right]_1^e = e-1$ then $I_1 = e - I_0 = 1$ and $I_2 = e - 2I_1 = e-2$; therefore $\int_1^e f(x)\,dx = e-3$.

Finally , $S = 3-e$ *units of area* ; that is $S = 12-4e \ cm^2$.