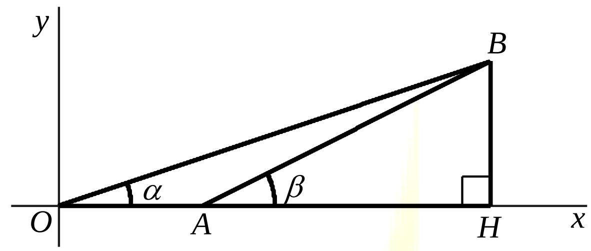
UNIVERSITY LEBANESE

FACULTY OF ENGINEERING

الجامعة اللبنانية
كلية الهندسة

Entrance Exam 2012 - 2013 | **Mathematics** | **Duration : 3 hours**
July 07 , 2012

The distribution of grades is over 25



I- (1.5 pt) The plane is referred to a direct orthonormal system of origin O. Consider the points $A(1\,;\,0)$ and $B(3\,;\,1)$.

Let z_1 be the affix of $\overrightarrow{OB}$ and z_2 that of $\overrightarrow{AB}$.

1- Determine an argument of $z_1\,z_2$ in terms of α and β .

2- Determine the algebraic form of each of z_1 , z_2 and $z_1\,z_2$.

3- Deduce the value of the sum $\alpha+\beta$.

II- (3.5 pts) Consider the sequence (I_n) defined , for $n\geq 1$, by $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx$.

1- Calculate I_1 and I_2 .

2- Prove that , for all $n\geq 1$, $I_n+I_{n+2}=\dfrac{1}{n+1}$. Deduce $\int_{-\frac{\pi}{4}}^{0}\tan^3 x\,dx$ and $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\tan^4 x\,dx$.

3- a) Prove that , for all $n\geq 1$, $I_n\geq 0$. Deduce that $I_n\leq\dfrac{1}{n+1}$.

b) Prove that the sequence (I_n) is decreasing . Deduce that $I_n\geq\dfrac{1}{2(n+1)}$.

4- Prove that the sequence (I_n) is convergent and calculate its limit .

III- (5.5 pts) ***A-*** The complex plane is referred to a direct orthonormal system .

1- Solve , in the set of complex numbers , the equation $z^2-2(1+\cos 2\alpha)z+2(1+\cos 2\alpha)=0$ where $\alpha\in\left]0\,;\,\frac{\pi}{2}\right[$.

2- Determine the exponential form of the complex number $q=1+\cos 2\alpha+i\sin 2\alpha$.

3- Consider the complex number $z=\dfrac{4}{q^2}$ and designate its image by M .

a) Determine the exponential form of z in terms of α . Deduce that $z=1-\tan^2\alpha-2i\tan\alpha$.

b) Prove that , as α traces $\left]0\,;\,\frac{\pi}{2}\right[$, the set of M is a part of a parabola to be determined .

B- Consider the parabola (P) of equation $y^2 = 4 - 4x$.

1- Determine the vertex S of (P) and draw (P) in an orthonormal system $(O\,;\vec{i}\ ,\vec{j}\,)$.

2- Let A and B be two points of (P) distinct from S such that (SA) and (SB) are perpendicular .
Let $2a$ and $2b$ be the respective ordinates of A and B .

a) Determine b in terms of a .

b) Prove that , as a varies in IR^*, (AB) passes through the fixed point I such that $\overrightarrow{OI} = -3\overrightarrow{OS}$.

c) Let K be the symmetric of S with respect to (AB) . Prove that , as a varies in IR^*, K varies on a fixed circle to be determined .

d) Determine the abscissa of the point of intersection L of the tangents to (P) at A and B respectively and prove that , as a varies in IR^*, L varies on a fixed straight line to be determined .

IV- (3.5 pts) A beginner in darts executes successive throws . We know that :

- The probability that he hits the target at the first throw is equal to 0.5 .
- If he hits the target at a certain throw , the probability that he hits the target at the next throw is equal to 0.4 .
- If he misses the target at a certain throw , the probability that he misses it at the next throw is equal to 0.8 .

For all natural numbers $n \geq 1$, consider the events A_n : " the player hits the target at the n^{th} throw " and B_n : " the player misses the target at the n^{th} throw " and let $p_n = p(A_n)$.

1- For all $n \geq 1$, determine $p(A_{n+1}/A_n)$ and $p(A_{n+1}/B_n)$.

2- Prove that , for all $n \geq 1$, $p_{n+1} = 0.2(1 + p_n)$.

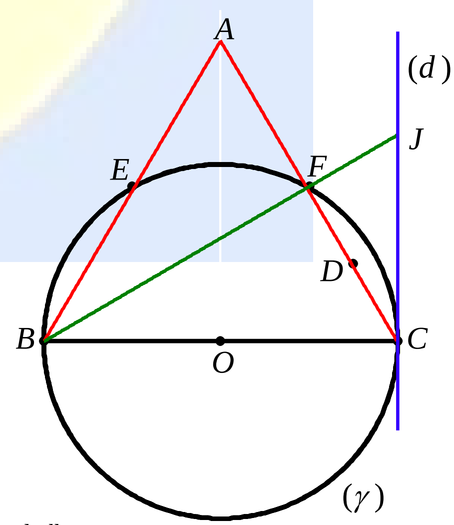
3- Consider the sequence (V_n) defined , for all $n \geq 1$, by $V_n = p_n - 0.25$.

a) Prove that (V_n) is a geometric sequence whose common ratio and first tem are to be determined .

b) Calculate V_n and then p_n in terms of n and determine $\lim\limits_{n \to +\infty} p_n$.

4- Knowing that the player hits the target at the second throw , calculate the probability that he missed it at the the first throw .

V- (4 pts) Given a circle (γ) of diameter $[BC]$, $BC = 2$ and center O .
(d) is the tangent to (γ) at C ; the triangle ABC is direct and equilateral of center G . (AB) cuts (γ) at E , (AC) cuts (γ) at F and (BF) cuts (d) at J . D is the mid point of $[CF]$.



Let S be the similitude of center C that transforms F into J .

1- a) Determine $S(D)$ and $S(A)$.

b) Determine the ratio and an angle of S .

c) Prove that $S(E) = A$ and that $S(O) = G$.

2- Let (γ') be the image of the circle (γ) by S.

a) Prove that (γ') is the circle circumscribed about the triangle ABC.

b) Prove that J belongs to (γ'). Draw (γ').

c) Calculate the area of the circle (γ').

VI- (7 pts) ***A-*** Consider the differential equation $(E):\ xy'+(1-x)y+x=0$ where y is a function defined on $IR-\{0\}$.

1- Prove that the function z such that $z=xy-x$ is the general solution of the differential equation $(I):\ z'-z=-1$.

2- Solve the equation (I) and determine the general solution of the equation (E).

3- Determine the particular solution of the equation (E) that has a finite limit at 0.

B- Consider the function f defined on IR such that $f(0)=0$ and , if $x\neq 0$, $f(x)=\dfrac{x+1-e^{x}}{x}$.

Let (C) be the representative curve of f in an orthonormal system $(O\ ;\vec{i}\ ,\ \vec{j})$.

1- Let g be the function defined on IR by $g(x)=x-e^{x}$.

Calculate $g(0)$ and $g'(0)$. Deduce $\lim\limits_{x\to 0}\dfrac{x+1-e^{x}}{x}$ and prove that f is continuous at 0.

2- a) We know that , for all $x\neq 0$, $-x^{2}-\dfrac{1}{2}x<f(x)<-\dfrac{1}{2}x$. Deduce that f is differentiable at 0.

b) Determine an equation of the tangent (δ) to (C) at the origin O .

3- Let h be the function defined on IR by $h(x)=(1-x)e^{x}$.

Set up the table of variations of h and prove that , for all x in $IR-\{0\}$, $h(x)<1$.

4- a) Prove that , for x in $IR-\{0\}$, $f'(x)=\dfrac{h(x)-1}{x^{2}}$ and set up the table of variations of f .

b) Determine the asymptote (d) to (C) at $-\infty$. Draw (d), (δ) and (C). (***Unit : 2 cm***)

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Exercise 1

1- The figure shows that $(\overrightarrow{u}\ ;\overrightarrow{OC}) = \alpha\ \ (2\pi)$ and $(\overrightarrow{u}\ ;\overrightarrow{AC}) = \beta\ \ (2\pi)$ then α is an argument of z_1 and β is an argument of z_2 ; therefore $\alpha+\beta$ is an argument of $z_1 z_2$.

2- $\overrightarrow{OC}(3\,;1)$ then $z_1 = 3+i$ and $\overrightarrow{AC}(2\,;1)$ then $z_2 = 2+i$; therefore $z_1 z_2 = (3+i)(2+i) = 5+5i$.

3- $z_1 z_2 = 5+5i = 5\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right) = 5\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ then $\frac{\pi}{4}$ is also an argument of $z_1 z_2$.

$\alpha+\beta$ and $\frac{\pi}{4}$ are two arguments of the complex number $z_1 z_2$ then there exists an algebraic integer k such that $\alpha+\beta = \frac{\pi}{4}+2k\pi$.

$0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$ then $0<\alpha+\beta<\pi$; therefore $\alpha+\beta = \frac{\pi}{4}$.

Exercise 2

1- a) ▪ $I_1 = \int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = -\int_0^{\frac{\pi}{4}} \frac{(\cos x)'}{\cos x}\,dx = -\left[\ell n\left|\cos x\right|\right]_0^{\frac{\pi}{4}} = -\ell n\frac{1}{\sqrt{2}} + \ell n 1 = \ell n\sqrt{2}$.

▪ $I_2 = \int_0^{\frac{\pi}{4}} \tan^2 x\,dx = \int_0^{\frac{\pi}{4}} (1+\tan^2 x - 1)\,dx = \left[\tan x - x\right]_0^{\frac{\pi}{4}} = 1 - \frac{\pi}{4}$.

2- a) $I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \left(\tan^n x + \tan^{n+2} x\right)dx = \int_0^{\frac{\pi}{4}} \tan^n x\left(1+\tan^2 x\right)dx = \left[\frac{\tan^{n+1} x}{n+1}\right]_0^{\frac{\pi}{4}} = \frac{1}{n+1}$.

b) ▪ The function $x \to \tan^3 x$ is odd then $\int_{-\frac{\pi}{4}}^{0} \tan^3 x\,dx = -\int_0^{\frac{\pi}{4}} \tan^3 x\,dx = -I_3 = -\frac{1}{2} + I_1 = -\frac{1}{2} + \ell n\sqrt{2}$

▪ The function $x \to \tan^4 x$ is even then $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^4 x\,dx = 2\int_0^{\frac{\pi}{4}} \tan^4 x\,dx = 2I_4 = 2\left(\frac{1}{3} - I_2\right) = \frac{\pi}{2} - \frac{4}{3}$.

3- a) ▪ For all x in $[0\,;\frac{\pi}{4}]$, $\tan x \geq 0$ then , for all x in $[0\,;\frac{\pi}{4}]$ and for all $n \geq 1$, $\tan^n x \geq 0$.

$\tan^n x \geq 0$ and $0 < \frac{\pi}{4}$ then $I_n \geq 0$.

▪ Using the relation $I_n + I_{n+2} = \frac{1}{n+1}$ we find $I_n = \frac{1}{n+1} - I_{n+2}$ where $I_{n+2} > 0$ then $I_n < \frac{1}{n+1}$.

b) ▪ $I_{n+1} - I_n = \int_0^{\frac{\pi}{4}} (\tan^{n+1} x - \tan^n x)\,dx = \int_0^{\frac{\pi}{4}} \tan^n x\,(\tan x - 1)\,dx$ where , in the interval $[0\,;\frac{\pi}{4}]$, $\tan^n x \geq 0$

$\tan x \leq 1$ and $0 < \frac{\pi}{4}$ then $I_{n+1} - I_n \leq 0$; that is $I_{n+1} \leq I_n$ and (I_n) is decreasing .

▪ (I_n) is decreasing then $I_n + I_{n+2} \leq 2I_n$.

Using the relation $I_n + I_{n+2} = \frac{1}{n+1}$ we find $\frac{1}{n+1} \leq 2I_n$ then $I_n \geq \frac{1}{2(n+1)}$.

c) The sequence (I_n) is decreasing and is bounded from below by 0 then it is convergent .

3- ▪ (I_n) is decreasing and bounded from below by 0 then (I_n) is convergent .

▪ The limit ℓ should satisfy the relation $\ell + \ell = \lim\limits_{n\to+\infty} \frac{1}{n+1}$; therefore $2\ell = 0$ and $\ell = 0$.

OR $\frac{1}{2(n+1)} \le I_n < \frac{1}{n+1}$ where $\lim\limits_{n\to+\infty} \frac{1}{n+1} = \lim\limits_{n\to+\infty} \frac{1}{2(n+1)} = 0$ then $\lim\limits_{n\to+\infty} I_n = 0$.

Exercise 3

A- 1- For all real numbers α , $z^2 - 2(1+\cos 2\alpha) z + 2(1+\cos 2\alpha) = 0$ is a second degree equation whose discriminant is $\Delta' = (1+\cos 2\alpha)^2 - 2(1+\cos 2\alpha) = \cos^2 2\alpha - 1 = -\sin^2 2\alpha = (i \sin 2\alpha)^2$.

The solutions in C of the given equation are $z = 1+\cos 2\alpha + i\sin 2\alpha$ and $z = 1+\cos 2\alpha - i\sin 2\alpha$.

2- $q = 1+\cos 2\alpha + i\sin 2\alpha$ where $0 < \alpha < \frac{\pi}{2}$.

$q = 1+\cos 2\alpha + i\sin 2\alpha = 2\cos^2\alpha + 2i \sin\alpha\cos\alpha = 2\cos\alpha(\cos\alpha + i\sin\alpha) = 2\cos\alpha\, e^{i\alpha}$ where $\cos\alpha > 0$ since $0 < \alpha < \frac{\pi}{2}$; therefore , the exponential form of q is $q = 2\cos\alpha\, e^{i\alpha}$.

3- Consider the complex number $z = \frac{4}{q^2}$. Let M be the image of z .

a) $q^2 = 4\cos^2\alpha\, e^{i2\alpha}$ then $z = \frac{4}{q^2} = \frac{1}{\cos^2\alpha} e^{-i2\alpha}$.

$$z = \frac{1}{\cos^2\alpha} e^{-i2\alpha} = \frac{1}{\cos^2\alpha}\left(\cos 2\alpha - i\sin 2\alpha\right) = \frac{\cos 2\alpha}{\cos^2\alpha} - \frac{\sin 2\alpha}{\cos^2\alpha} i = \frac{2\cos^2\alpha - 1}{\cos^2\alpha} - \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha} i$$
$$= 2 - \left(1+\tan^2\alpha\right) - 2\tan\alpha\, i = 1 - \tan^2\alpha - 2\tan\alpha\, i .$$

b) The coordinates of M are $x = 1-\tan^2\alpha$ and $y = -2\tan\alpha$.

As α varies , the coordinates of M satisfy the relation $x = 1 - \frac{1}{4} y^2$ then M belongs to the parabola of equation $y^2 = -4(x-1)$.

As α traces the interval $]0\,;\frac{\pi}{2}[$ the ordinate y of M traces the interval $]-\infty\,;0[$.

Therefore the set of N is the part of the parabola lying below the axis of abscissas .

B- (P) is the parabola of equation $y^2 = 4-4x$; $y^2 = 4(1-x)$.

1- The vertex of (P) is the point $S(1\,;0)$.

Drawing (P) .

2- The abscissa of the point A of (P) with ordinate a such that $a \neq 0$ is equal to $1-a^2$ then $A(1-a^2\ ;\ 2a)$.

Similarly , $B(1-b^2\ ;\ 2b)$.

a) $\overrightarrow{SA}(-a^2\ ;\ 2a)$ and $\overrightarrow{SB}(-b^2\ ;\ 2b)$.

(SA) and (SB) are perpendicular if and only if $\overrightarrow{SA}.\overrightarrow{SB} = 0$; that is $a^2b^2 + 4ab = 0$; $ab(ab+4) = 0$

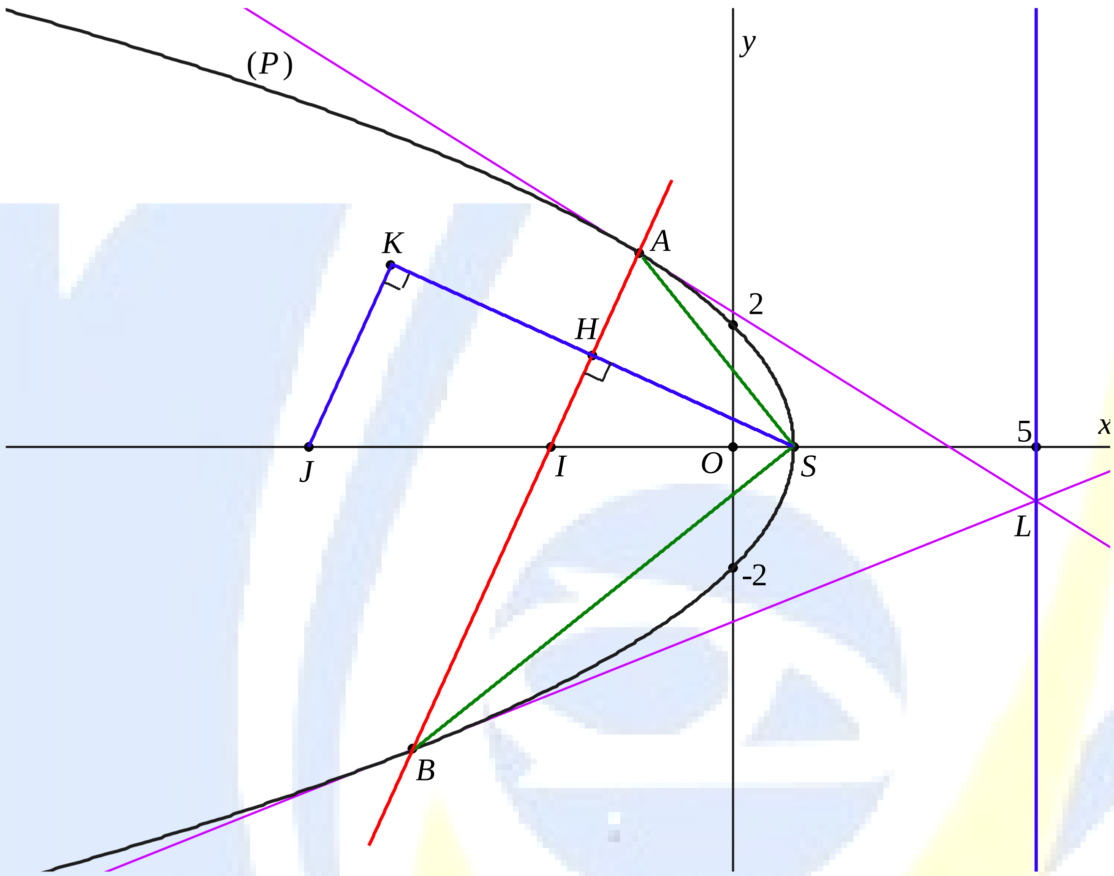
where $ab \neq 0$ then $ab + 4 = 0$ and $b = -\dfrac{4}{a}$.

b) The point I is such that $\overrightarrow{OI} = -3\overrightarrow{OS}$ then $I(-3\ ;\ 0)$.

$\det(\overrightarrow{IA}\ ;\ \overrightarrow{IB}) = \begin{vmatrix} 4-a^2 & 2a \\ 4-b^2 & 2b \end{vmatrix} = 8b - 2a^2b - 8a + 2ab^2$ with $b = -\dfrac{4}{a}$ then

$\det(\overrightarrow{IA}\ ;\ \overrightarrow{IB}) = -\dfrac{32}{a} + 8a - 8a + \dfrac{32}{a} = 0$.

Therefore , as a varies in IR^* , A , B and I are collinear then (AB) passes I .



c) Let H be the orthogonal projection of S on (AB) .

The angle $S\widehat{H}I$ is right with S and I are fixed then H varies on the circle (γ) of diameter $[SI]$.

The symmetric of S with respect to (AB) is the point K such that $\overrightarrow{SK} = 2\overrightarrow{SH}$ then K is the image of H by the dilation $h(S\ ;\ 2)$. Therefore K varies on the circle $(\gamma') = h((\gamma))$ with diameter $[SJ]$. where $J = h(I)$; $J(-7\ ;\ 0)$.

d) The equation $y^2 = 4 - 4x$ gives $2\,y\,y' = -4$ then the slope of the tangent to (P) at A is equal to $-\dfrac{1}{a}$.

An equation of the tangent (d_1) to (P) at A is $y = -\dfrac{1}{a}(x - 1 + a^2) + 2a$; $y = -\dfrac{1}{a}(x - 1 - a^2)$.

Similarly , an equation of the tangent (d_2) to (P) at B is $y = -\dfrac{1}{b}(x - 1 - b^2)$.

The abscissa of the point of intersection L of (d_1) and (d_2) is the solution of the equation

$\frac{1}{a}(x-1-a^2)=\frac{1}{b}(x-1-b^2)$; $b(x-1-a^2)=a(x-1-b^2)$; $(a-b)x=a-b+ab^2-ab^2$; $x=1-ab=5$.

As a varies in IR^*, the point L that has a constant abscissa varies of the straight line of equation $x=5$.

Exercise 4

1- ▪ It is given that , if he hits the target at a certain throw , the probability that he hits it at the next throw is equal to 0.4 ; therefore $p(A_{n+1}/A_n)=0.4$.

▪ If he misses the target at a certain throw , the probability that he misses it at the next throw is equal to 0.8 ; therefore $p(B_{n+1}/B_n)=0.8$.

Hence $p(A_{n+1}/B_n)=p(\overline{B_{n+1}}/B_n)=1-p(B_{n+1}/B_n)=0.2$.

2- For all $n\geq 1$, $p_{n+1}=p(A_{n+1})=p(A_{n+1}\cap A_n)+p(A_{n+1}\cap B_n)$.

$$=p(A_n)\times p(A_{n+1}/A_n)+p(B_n)\times p(A_{n+1}/B_n)$$

$$=0.4p_n+(1-p_n)\times 0.2=0.2p_n+0.2=0.2(1+p_n)\ .$$

3- The sequence (V_n) is defined , for all $n\geq 1$, by $V_n=p_n-0.25$.

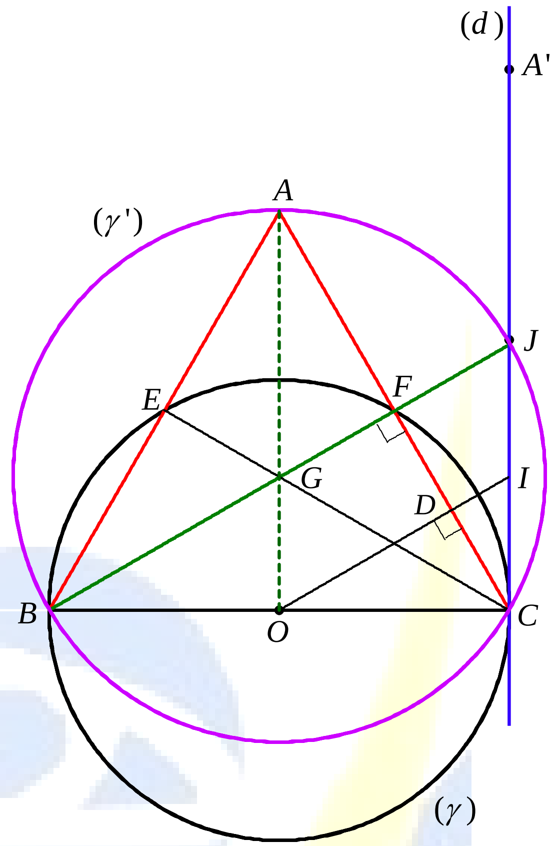
a) $V_{n+1}=p_{n+1}-0.25=0.2p_n+0.2-0.25=0.2p_n-0.05=0.2\left(p_n-0.25\right)=0.2V_n$. Therefore (V_n) is a geometric sequence whose common ratio is $r=0.2$ and first tem $V_1=p_1-0.25=0.5-0.25=0.25$.

b) $V_n=V_1\times r^{n-1}=0.25\times(0.2)^{n-1}$ and $p_n=0.25\times(0.2)^{n-1}+0.25$.

Since $0<0.2<1$, $\lim\limits_{n\to+\infty}(0.2)^{n-1}=0$; therefore $\lim\limits_{n\to+\infty}p_n=0.25$.

4- The required probability is

$$p(B_1/A_2)=\frac{p(B_1\cap A_2)}{p(A_2)}=\frac{p(B_1)\times p(A_2/B_1)}{p_2}=\frac{0.5\times 0.2}{0.2p_1+0.2}=\frac{0.5}{p_1+1}=\frac{0.5}{0.5+1}=\frac{1}{3}.$$



Exercise 5

S is the similitude of center C such that $S(F) = J$.

1- a) ▪ $S(F) = J$ then $S([CF]) = [CJ]$.

D is the mid point of $[CF]$ then $S(D)$ is the mid point I of $[CJ]$.

▪ Let $S(A) = A'$.

The triangle ABC is equilateral and (BF) is a height in this triangle ; therefore F is the mid point of $[CA]$.

$S(C) = C,\ S(F) = J,\ S(A) = A'$.

A is the symmetric of C with respect to F then A' is the symmetric of C with respect to J .

$S(F) = J$ and

$(\overrightarrow{CF}\ ;\ \overrightarrow{CJ}) = (\overrightarrow{CB}\ ;\ \overrightarrow{CJ}) - (\overrightarrow{CB}\ ;\ \overrightarrow{CF}) = -\frac{\pi}{2} + \frac{\pi}{3} = -\frac{\pi}{6}\ (2\pi)$ then $-\frac{\pi}{6}$ is an angle of S .

The ratio of S is $\frac{CJ}{CF} = \frac{1}{\cos\frac{\pi}{6}} = \frac{2}{\sqrt{3}}$. $S = S(C\ ;\ \frac{2}{\sqrt{3}}\ ;\ -\frac{\pi}{6})$.

b) ▪ In the triangle CEA we have $(\overrightarrow{CE}\ ;\ \overrightarrow{CA}) = -\frac{\pi}{6}\ (2\pi)$ and $\frac{CA}{CE} = \frac{1}{\cos\frac{\pi}{6}} = \frac{2}{\sqrt{3}}$, then $S(E) = A$.

▪ In the triangle COG we have $(\overrightarrow{CO}\ ;\ \overrightarrow{CG}) = -\frac{\pi}{6}\ (2\pi)$ and $\frac{CG}{CO} = \frac{1}{\cos\frac{\pi}{6}} = \frac{2}{\sqrt{3}}$, then $S(O) = G$.

2- The circle (γ') is the image of the circle (γ) by S .

a) (γ) is the circle of center O passing though E then its image (γ') is the circle of center $G = S(O)$ passing through $A = S(E)$. Hence , (γ') is the circle circumscribed about the triangle ABC .

b) F belongs to (γ) and $S(F) = J$ then J belongs to (γ') . Drawing (γ')

c) The radius of the circle (γ) is $OC=1$ then the radius of the circle (γ') which is $S((\gamma))$, is equal to $\frac{2}{\sqrt{3}}$ then its area is $\pi\left(\frac{2}{\sqrt{3}}\right)^2=\frac{4}{3}\pi$ *units of area* .

Exercise 6

A- $(E):\ xy'+(1-x)y+x=0$ where y is a function defined on $IR-\{0\}$.

1- If $z=xy-x$ then $z'=y+xy'-1$ and $z'-z=(y+xy'-1)-(xy-x)=xy'+(1-x)y+x-1=-1$.
Therefore z is the general solution of the differential equation (I) : $z'-z=-1$.

2- The general solution of the reduced equation $z'-z=0$ is $z=Ce^x$.
The general solution of the equation (I) is $z=Ce^x+1$.
The general solution of the equation (E) is $y=\frac{x+z}{x}$; that is $y=\frac{x+1+Ce^x}{x}$.

3- $\lim\limits_{x\to 0}(x+1+Ce^x)=1+C$.
If $C\neq -1$ then , $\lim\limits_{x\to 0} y=\pm\infty$.
If $C=-1$ then , $\lim\limits_{x\to 0^+} y=\lim\limits_{x\to 0^+}\frac{1-e^x}{1}=0$.
The particular solution of the equation (1) that has a finite limit at 0 is $y=\frac{x+1-e^x}{x}$.

B- The function f defined on IR by $\begin{cases} f(x)=\dfrac{x+1-e^x}{x} & if\ \ x\neq 0 \\ f(0)=0 \end{cases}$.

1- The function g is defined on IR by $g(x)=x-e^x$.
$g(0)=-1$; $g'(x)=1-e^x$ then $g'(0)=0$.
$\lim\limits_{x\to 0}\frac{x+1-e^x}{x}=\lim\limits_{x\to 0}\frac{g(x)-g(0)}{x-0}=g'(0)=0$.
$\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\frac{x+1-e^x}{x}=0=f(0)$; therefore f is continuous at 0.

2- Given that , for all $x\neq 0$, $-x^2-\frac{1}{2}x<f(x)<-\frac{1}{2}x$.

- For all $x<0$, $-\frac{1}{2}<\frac{f(x)}{x}<-x-\frac{1}{2}$ with $\lim\limits_{x\to 0^-}(-x-\frac{1}{2})=-\frac{1}{2}$ then $\lim\limits_{x\to 0^-}\frac{f(x)}{x}=-\frac{1}{2}$.

- For all $x>0$, $-x-\frac{1}{2}<\frac{f(x)}{x}<-\frac{1}{2}$ with $\lim_{x\to 0^+}(-x-\frac{1}{2})=-\frac{1}{2}$ then $\lim_{x\to 0^+}\frac{f(x)}{x}=-\frac{1}{2}$.

$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=-\frac{1}{2}$ which is finite then f is differentiable at 0 and $f'(0)=-\frac{1}{2}$.

An equation the tangent (δ) to (C) at the origin is $y=-\frac{1}{2}x$.

3- The function h is defined on IR by $h(x)=(1-x)e^x$.

$\lim_{x\to-\infty}h(x)=\lim_{x\to-\infty}(e^x-xe^x)=0$ and $\lim_{x\to+\infty}h(x)=-\infty$

$h'(x)=-xe^x$.

Table of variations of h

x	$-\infty$		0		$+\infty$
$h'(x)$		$+$	0	$-$	
$h(x)$	0	↗	1	↘	$-\infty$

The table of variations of h shows that $h(0)=1$ is the absolute maximum of h and then , for all x in $IR-\{0\}$, $h(x)<1$.

4- a) For x in $IR-\{0\}$, $f(x)=\frac{x+1-e^x}{x}$ then $f'(x)=\frac{(1-x)e^x-1}{x^2}=\frac{h(x)-1}{x^2}$.

$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\left(1+\frac{1}{x}-\frac{e^x}{x}\right)=1.$

$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\left(1+\frac{1}{x}-\frac{e^x}{x}\right)=-\infty$

For x in $IR-\{0\}$, $f'(x)=\frac{h(x)-1}{x^2}<0$ since $h(x)<1$.

Table of variations of f .

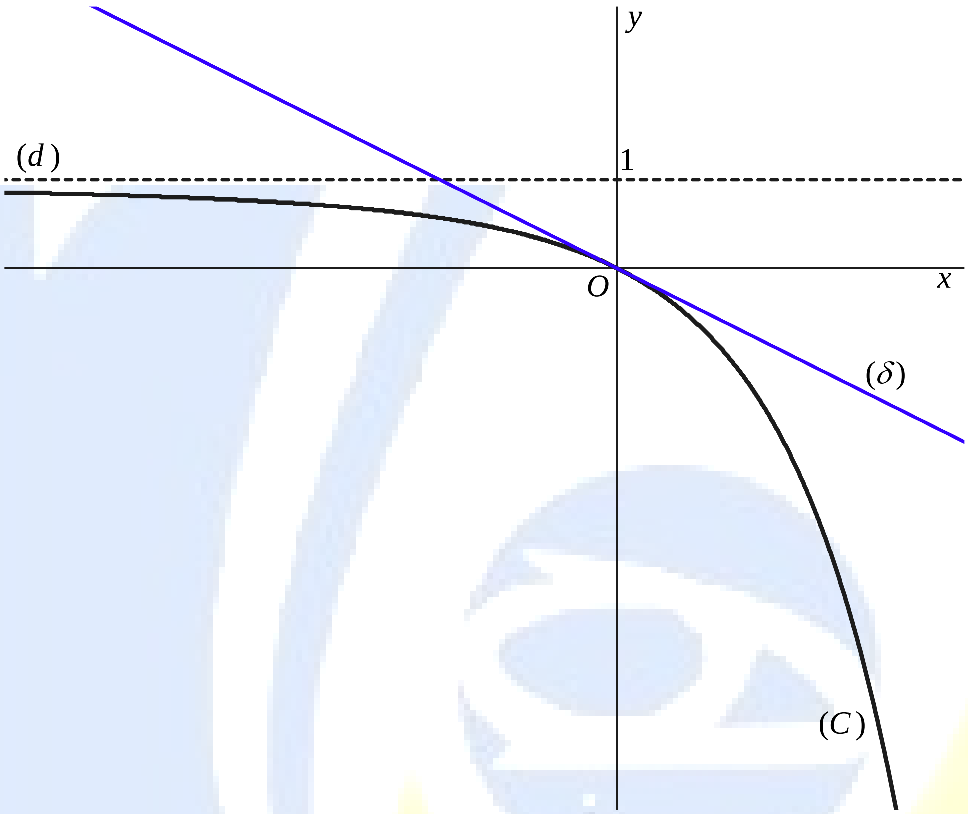
x	$-\infty$		$+\infty$
$f'(x)$		$-$	
$f(x)$	1	↘	$-\infty$

c) $\lim_{x\to-\infty}g(x)=1$ then the straight line (d) is asymptote to (C) at $-\infty$.

$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x+1-e^x}{x^2}=\lim_{x\to+\infty}\left(\frac{1}{x}+\frac{1}{x^2}-\frac{e^x}{x^2}\right)=-\infty$ since $\lim_{x\to+\infty}\frac{e^x}{x^2}=+\infty$.

Therefore (C) has at $+\infty$ an asymptotic direction parallel to the axis of ordinates .

Drawing (d), (δ) and (C). (***Unit : 2 cm***)


y
(d)
1
O
x
(δ)
(C)