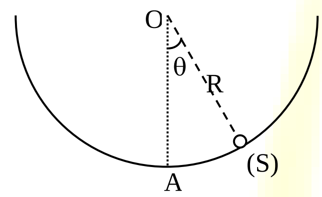
LEBANESE UNIVERSITY

FACULTY OF ENGINEERING

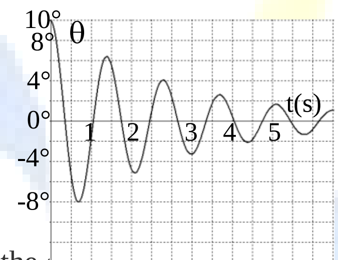
الجامعة اللبنانية
كلية الهندسة

Entrance Exam 2012 – 2013 **<u>PHYSICS</u>** **Duration: 1H**
8 JULY 2012

Exercise I: [12 pts] Study of the motion of a particle



Consider a hollow circular slide (C) of radius R = 50 cm and located in a vertical plane. A particle (S), of mass m = 20 g, can slide on the inner surface of (C). Initially, (S) is at A, its position of stable equilibrium. We shift (S), in the positive direction, by an angle $\theta_0 = 10^o$, then we release it from rest at the instant $t_0 = 0$. At an instant t, its angular elongation is θ and its angular velocity is $\dot{\theta}$. The horizontal plane through A is the reference level for the gravitational potential energy. Take $g = 10\ m/s^2$; $\sin\theta \approx \theta$ (rad).



1. We neglect the forces of friction.
   a) Determine, at the instant t, the mechanical energy of the system ((S), Earth).
   b) Derive the second order differential equation in θ that describes the oscillations of (S).
   c) Deduce the value of the proper (natural) period of these oscillations.
   d) Determine the time equation of motion.
2. In fact, considering the same initial conditions, (S) undergoes, at an instant t, the force of friction $\vec{f} = -\lambda\vec{V}$, where λ is a positive constant.
   a) Determine the expression of the power of the frictional force at the instant t. Deduce that the differential equation that governs the motion of (S) is written as: $\ddot{\theta} + \frac{\lambda}{m}\dot{\theta} + \frac{g}{R}\theta = 0$.
   b) The solution of this differential equation is of the form: $\theta(t) = A\exp(-\lambda t/(2m))\cos(\omega t-\varphi)$.
   Take $\delta = \theta(t+T)/\theta(t)$, where T is the pseudo-period. Determine the expression of δ and deduce the value of λ.

Exercise II: [15 pts] Why is the sky blue?

In 1904, Sir J.J Thomson proposed a model for the hydrogen atom, in which the electron of mass m, located at M, is elastically linked to its fixed nucleus located at O. The atom is thus reduced to an elastic pendulum (m, k), the electron of mass m undergoing the force $\vec{F_e} = -k\overrightarrow{OM}$ where $\overrightarrow{OM} = x\vec{i}$ and where O is its stable equilibrium position. The electron may thus move along $\vec{i}$. Given: $m = 9.1\cdot10^{-31}$ kg, k = 100 N/m and we neglect the weight of the electron.
1. a) Neglecting frictional forces, derive the differential equation of motion of the oscillator.
   b) Deduce the expression of the proper angular frequency ω_0 and that of the proper period T_0 of the oscillator.
   c) Calculate the values of ω_0 and T_0.
2. A luminous wave, issued from the Sun, is characterized by an electric field $\vec{E} = E_0\cos(\omega t+\varphi)\vec{i}$, where ω belongs to the interval $\omega_{red} \leq \omega \leq \omega_{blue}$, these two extreme radiations having the following wavelengths in vacuum: $\lambda_{red} = 0.800\ \mu m$ and $\lambda_{blue} = 0.400\ \mu m$. We intend to study the action of this wave on the electron of an atom of the atmosphere, using the

Thomson model. The electron thus undergoes, at an instant t, the electric force $\vec{F'} = -e\vec{E} = -e\,E_0\cos(\omega t+\varphi)\,\vec{i}$ and, in addition, a frictional force of the form $\vec{F} = -h\,v\,\vec{i}$ where $v = \frac{dx}{dt}$. Given: $e = 1{,}6.10^{-19}$ C; $h = 10^{-20}$ kg/s.

a) Show that the differential equation in x is of the form: $\ddot{x} + B\dot{x} + \omega_0^2\,x = -D\cos(\omega t+\varphi)$.

b) Determine the expressions of the positive constants B and D and calculate the value of B.

c) Calculate the value of ω_{red} and that of ω_{blue}.

3. The solution of this differential equation, in steady state, is of the form $x = A\cos(\omega t)$. By giving ωt two particular values, determine the expression of A in terms of ω.

4. By giving ω the considered limiting values, show that the expression of A can be reduced to: $A \approx \frac{e.E_0}{m(\omega^2{}_0 - \omega^2)}$.

5. Knowing that the electron emits, in all directions, an electromagnetic radiation whose average power is proportional to the square of the amplitude of its acceleration,

a) Give the expression for the average power P_{ave} in terms of e, m, E_0, ω and ω_0.

b) By comparing the two average powers P_{red} and P_{blue}, explain why the sky is blue.

Exercise III: [15 pts] Rubidium-Strontium dating

Some granitic rocks, during their crystallization, have detained an amount of rubidium ${}^{87}_{37}Rb$, a radioactive isotope of rubidium, of radioactive constant $\lambda = 1{,}42\times10^{-11}$ year^{-1}, and another amount of strontium formed of stable isotopes (${}^{87}_{38}Sr$) and (${}^{86}_{38}Sr$). A ${}^{87}_{37}Rb$ nucleus decays into a ${}^{87}_{38}Sr$ nucleus.

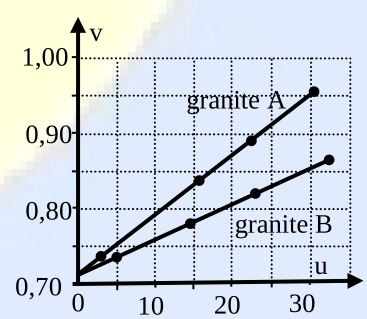
1. Give, with justification, the type of the decay of a ${}^{87}_{37}Rb$ nucleus.

2. Calculate the radioactive half-life $t_{½} = T$ of the rubidium 87 sample.

3. $N({}^{87}_{37}Rb)$ and $N_0({}^{87}_{37}Rb)$ are respectively the number of rubidium atoms present at the current instant t and that of the atoms that were present at the instant $t_0 = 0$, instant of rock formation. Show that the number $N^*({}^{87}_{38}Sr)$ of strontium atoms formed from the instant t_0 until the instant t has the expression: $N^*({}^{87}_{38}Sr) = N({}^{87}_{37}Rb)\,(e^{\lambda t} - 1)$.

4. $N_0({}^{87}_{38}Sr)$ is the initial number of strontium-87 nuclei present in the sample. Give the expression $N({}^{87}_{38}Sr)$ of the total number of these nuclei present in the sample at the current instant t in terms of $N({}^{87}_{37}Rb)$, $N_0({}^{87}_{38}Sr)$, λ and t.

5. By measuring experimentally the ratios $u = \frac{N({}^{87}_{37}Rb)}{N({}^{86}_{38}Sr)}$ and $v = \frac{N({}^{87}_{38}Sr)}{N({}^{86}_{38}Sr)}$ in the minerals of two different granitic rocks (granite A, granite B), we obtain the adjacent two graphs.



a) Why was the ${}^{86}_{38}Sr$ isotope used as a reference?

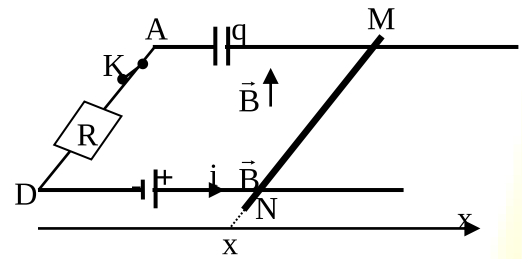
b) Show that we can write: $v = au + b$, where: $a = (e^{\lambda t} - 1)$.

c)i) Determine the value of **a** for each of the two granitic rocks.

ii) Deduce the approximate age of each of the two rocks.

d) Why didn't we use the carbon-14 of half-life of 5730 years for dating this rock?

Exercise IV: [18 pts] Charging a capacitor and motion of a rod



The circuit of the adjacent figure consists of two horizontal and parallel Laplace's rails connected to an ideal generator of emf E = 6 V, a capacitor (C) of capacitance C = 0.1 F and a resistor of resistance R = 5 Ω. The rails, being horizontal and separated by a distance ℓ = 10 cm, are placed in an upward vertical magnetic field and of magnitude B = 1.0 T.
A metallic rod MN, of mass m = 0.10 kg, can move without friction on the rails while remaining perpendicular to these rails. The two rails and the rod are of negligible resistance.
At the instant $t_0 = 0$, (C) being discharged, we close K. At an instant t, the circuit carries a current i, (C) is charged by q and has, across its terminals, the voltage $u_{MA} = u_C$. MN, located by its x-coordinate and undergoing the action of the Laplace's force, has a velocity $\vec{V}$ of algebraic value $V = \dfrac{dx}{dt}$ The circuit is thus oriented in the direction of i.

1. a) Give the direction of $\vec{F}$ and its magnitude F as a function of the current i.
   b) Show that the expression of the voltage across the terminals M and N of the rod is then written as $\mathbf{u}_{NM} = + B\ell V$.
2. a) Applying Newton's second law, show that $V = k\, u_C$, and determine the positive constant k.
   b) Applying the law of addition of voltages, derive the differential equation: $E = RC\dfrac{du_C}{dt} + \left(\dfrac{B^2\ell^2C + m}{m}\right)u_C$.

3-a) The solution of this equation is of the form $u_C = a - b\cdot e^{-t/\tau}$. Determine the values of the constants a, b and τ.
   b) Deduce the expressions, as a function of time t, of V and i.
   c) Determine x as a function of time t knowing that, at the instant $t_0 = 0$, $x_0 = 0$.
   d) i) Determine the instant t_1 at which the steady state is practically reached.
      ii) Determine the charge Q of (C), the abscissa x_1 of MN and the nature of motion of the rod starting from t_1.

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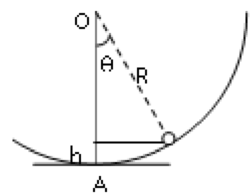
Exercise I :

1) a) M.E(t) = $\frac{1}{2}I\theta'^2 + mgh = \frac{1}{2}mR^2\theta'^2 + mgR(1-\cos\theta)$ 1 ½



b) Friction forces are negligible $\Rightarrow$ M.E(t) = constant $\Rightarrow \frac{dM.E}{dt} = 0$

$\Rightarrow mR^2\theta'\theta'' + mgR(\theta'\sin\theta) = 0$; $\theta' \neq 0 \Rightarrow R\theta'' + g\sin\theta = 0$

For small θ, $\sin\theta \approx \theta$ (in rad) $\Rightarrow \theta'' + \frac{g}{R}\theta = 0$ 2

c) The differential equation is of the form: $\theta'' + \omega_0^2\theta = 0 \Rightarrow \omega_0^2 = \frac{g}{R}$

$\Rightarrow T_0 = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{0.5}{10}} = 1.41$ s 1

d) The time equation of motion $\theta = \theta_m\cos(\omega_0 t + \varphi)$

and $\theta' = -\ \omega_0\theta_m\sin(\omega_0 t + \varphi)$

For t = 0 : $\theta = -\theta_m\cos(\varphi) = \theta_0$; 1 ½

and $\theta' = -\ \omega_0\theta_m\sin(\varphi) = 0 \Rightarrow \varphi = 0$ or π(rad)

For $\varphi = \pi \Rightarrow \theta_m = -\ \theta_0$;

and for $\varphi = 0 \Rightarrow \theta_m = \theta_0 \Rightarrow \varphi = \pi$ is rejected $\Rightarrow \theta = \theta_0\cos(\omega_0 t)$

2) a) $P = \vec{f}\cdot\vec{v} = -\lambda\,\vec{v}^2 = -\lambda v^2 = -\lambda R^2\theta'^2$ 1

The differential equation describing the motion of (S) is given by:

$$\frac{dM.E}{dt} = P \Rightarrow mR^2\theta'\theta'' + mgR(\theta'\sin\theta) = -\lambda\,R^2\theta'^2$$

1 ½

$$\Rightarrow mR^2\theta'\theta'' + \lambda R^2\theta'^2 + mgR(\theta'\sin\theta) = 0.$$

$$\Rightarrow R\theta'\theta'' + \frac{\lambda}{m}R\theta' + g(\theta'\theta) = 0 \Rightarrow \theta'' + \frac{\lambda}{m}\theta' + \frac{g}{R}\theta = 0$$

b) The coefficient $\delta = \dfrac{\theta(t+T)}{\theta(t)} \Rightarrow \delta = \dfrac{A\,e^{\frac{-\lambda(t+T)}{2m}}\cos[\omega(t+T)-\varphi]}{A\,e^{\frac{-\lambda t}{2m}}\cos[\omega t-\varphi]}$

$$\Rightarrow\ \delta = \frac{e^{\frac{-\lambda(t+T)}{2m}}}{e^{\frac{-\lambda t}{2m}}} = e^{\frac{-\lambda T}{2m}}\ ;\ \delta = \text{constant}\ \forall\ t.$$ 1 ½

$$\Rightarrow \delta = \frac{6.3}{10} = 0.63$$ 1

$$\Rightarrow \ell n(\delta) = -0.462 = -\frac{\lambda T}{2m}$$

$$\Rightarrow \lambda = \frac{0.462\times 2m}{T} = 0.013\ \text{kg/s}$$ 1

Exercise II:

1) a) No friction, conservation of mechanical energy:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Derive both sides with respect to time: mx'x" + kx'x = 0; x' ≠ 0. 1 ½

$$\Rightarrow x'' + \frac{k}{m}x = 0,$$ is the differential equation

b) The general form of the differential equation is: $x'' + \omega_0^2 x = 0$,

with ω_0 the proper (natural) angular frequency: ½

$\omega_0 = \sqrt{\frac{k}{m}}$ ½ and the proper (natural) period T_0 : $T_0 = 2\pi\sqrt{\frac{m}{k}}$. ½

c) The value of ω_0 : $\omega_0 = \sqrt{\frac{100}{9.1\times10^{-31}}} = 1.048\times10^{16}$ rad/s ½

and $T_0 = 5.994\times10^{-16}$ s. ½

2) a) According to Newton's second law: $\sum\vec{F} = \frac{d\vec{P}}{dt} = m\frac{d\vec{v}}{dt} = m\,x''\,\vec{i}$ 1

$m\,x''\,\vec{i} = -h\,x'\,\vec{i} - kx\,\vec{i} + \vec{F}' = -hx'\,\vec{i} + kx\,\vec{i} - e\,E_0\cos(\omega t+\varphi)\,\vec{i}$

$$x'' + \frac{h}{m}x' + \frac{k}{m}x = -\frac{eE_0}{m}\cos(\omega t+\varphi);$$ 1

$x'' + B\,x' + \omega_0^2\,x = -D\cos(\omega t+\varphi)$ with : $\omega_0^2 = \frac{k}{m}$.

b) $B = \frac{h}{m}$ ½ and $D = \frac{eE_0}{m}$ ½ . $B = \frac{10^{-20}}{9.1\times10^{-31}} = 1.10\cdot10^{10}\ s^{-1}$. ½

c) $\omega = \frac{2\pi c}{\lambda} \Rightarrow \omega_{red} = \frac{2\pi\times3\times10^8}{0.8\times10^{-6}} = 2.36\times10^{15}$ rad/s 1

and $\omega_{blue} = \frac{2\pi\times3\times10^8}{0.4\times10^{-6}} = 4.71\times10^{15}$ rad/s ½

3) $x' = -A\omega\sin(\omega t)$ and $x'' = -A\omega^2\cos(\omega t)$. By replacing each variable by its expression in the differential equation, we obtain:

$$-A\omega^2\cos(\omega t) - B\,A\omega\sin(\omega t) + \omega_0^2 A\cos(\omega t) = -\frac{eE_0}{m}\cos(\omega t+\varphi)$$

$$-A\omega^2\cos(\omega t) - B\,A\omega\sin(\omega t) + \omega_0^2 A\cos(\omega t) = -D\cos(\omega t+\varphi) \quad 1$$

For $\omega t = 0 \Rightarrow -A\omega^2 + \omega_0^2 A = -D\cos(\varphi)$ ½

For $\omega t = \frac{\pi}{2} \Rightarrow -BA\omega = -D\cos(\frac{\pi}{2}+\varphi) = D\sin(\varphi)$ ½

$D^2\cos^2(\varphi) + D^2\sin^2(\varphi) = D^2 = A^2[(\omega^2 - \omega_0^2)^2 + B^2\omega^2]^2$

$$A = \frac{D}{\sqrt{B^2\omega^2 + (\omega^2-\omega_0^2)^2}} \quad 1$$

4) For the two extreme radiations $\omega < \omega_0$, as well $B\omega \ll (\omega_0^2 - \omega^2)$

$$\Rightarrow A \approx \frac{D}{(\omega_0^2-\omega^2)}; \text{ Thus } A \approx \frac{eE_0}{m(\omega_0^2-\omega^2)}. \quad 1$$

5) a) The square of amplitude of the acceleration is:

$$(A_{acc})^2 = [\omega^2 A]^2 \approx \left(\frac{\omega^2 eE_0}{m(\omega_0^2-\omega^2)}\right)^2. \quad ½$$

Thus the average power $P_{ave} \approx cte \times \left(\frac{\omega^2 eE_0}{m(\omega_0^2-\omega^2)}\right)^2$ ½

b) Thus: $P_{blue} \approx cte \times \left(\frac{\omega_{blue}^2 eE_0}{m(\omega_0^2-\omega_{blue}^2)}\right)^2$ ½

$P_{red} \approx cte \times \left(\frac{\omega_{red}^2 eE_0}{m(\omega_0^2-\omega_{red}^2)}\right)^2$

½

$$\Rightarrow \frac{P_{blue}}{P_{red}} \cong \left[\frac{\omega_{blue}^2(\omega_0^2-\omega_{red}^2)}{\omega_{red}^2(\omega_0^2-\omega_{blue}^2)}\right]^2 = 22.7 \Rightarrow \text{the sky is blue}$$

Exercise III:

1) ${}^{87}_{37}Rb \longrightarrow {}^{87}_{38}Sr + {}^{a}_{z}p \Rightarrow a = 0; z = -1 \Rightarrow {}^{a}_{z}p = {}^{0}_{-1}e$, the emission is β^-.

1

2) $T = \dfrac{\ell n2}{\lambda} = \dfrac{0.693}{1.42\times10^{-11}} = 4.88\times10^{10}$ years

1

3) We know that $N({}^{87}_{37}Rb) = N_0({}^{87}_{37}Rb)e^{-\lambda t} \Rightarrow N_0({}^{87}_{37}Rb) = N({}^{87}_{37}Rb)e^{\lambda t}$

½

Number of disintegrated Rb = number of Sr formed

½

½

$$\Rightarrow N^*({}^{87}_{38}Sr) = N_0({}^{87}_{37}Rb) - N({}^{87}_{37}Rb)$$

1

$$= N({}^{87}_{37}Rb)e^{\lambda t} - N({}^{87}_{37}Rb)$$

$$\Rightarrow N^*({}^{87}_{38}Sr) = N({}^{87}_{37}Rb)(e^{\lambda t} - 1)$$

4) $N({}^{87}_{38}Sr) = N^*({}^{87}_{38}Sr) + N_0({}^{87}_{38}Sr)$

$$= N({}^{87}_{37}Rb)(e^{\lambda t} - 1) + N_0({}^{87}_{38}Sr)$$

1

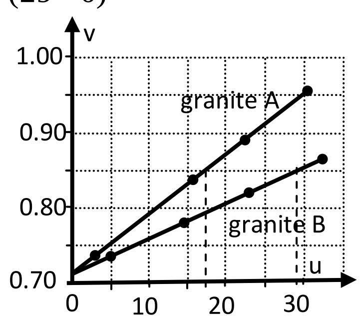
5) a) Since the isotope $^{86}_{38}Sr$ is stable and its number does not vary over time. 1

b) $\dfrac{N(^{87}_{38}Sr)}{N(^{86}_{38}Sr)} = \dfrac{N(^{87}_{37}Rb)(e^{\lambda t}-1)}{N(^{86}_{38}Sr)} + \dfrac{N_0(^{87}_{38}Sr)}{N(^{86}_{38}Sr)}$ 1 ½

Thus v = au + b $\Rightarrow$ a = $(e^{\lambda t} - 1)$ and b = $\dfrac{N_0(^{87}_{38}Sr)}{N(^{86}_{38}Sr)}$

c) i) For the granite A : $a_A = \dfrac{(0.85-0.715)}{(17-0)} = 7.94\times10^{-3}$. 1 1

For the granite B: $a_B = \dfrac{(0.85-0.715)}{(29-0)} = 4.65\times10^{-3}$ 1



ii) For A : $e^{\lambda t_A} - 1 = (7.94\times10^{-3}) \Rightarrow e^{\lambda t_A} = 1 + 7.94\times10^{-3}$ 1

$\Rightarrow e^{\lambda t_A} = 1.00794 \Rightarrow t_A = 5.57\times10^{8}$ years 1

For B : $e^{\lambda t_B} - 1 = (4.65\times10^{-3}) \Rightarrow e^{\lambda t_B} = 1 + 4.65\times10^{-3}$ 1

$\Rightarrow e^{\lambda t_B} = 1.00465 \Rightarrow t_B = 3.27\times10^{8}$ years

d) The carbon 14 (as other isotopes) is used to date samples whose ages do not exceed 10 T. Thus the carbon 14 dates at maximum 57000 years. 1

Exercise IV:

1) a) The force $\vec{F}$ is horizontal, directed to the right and of magnitude $F = iB\ell$. 1

b) The magnetic flux through the circuit is:

$\varphi = \vec{B}\cdot\vec{S} = \vec{B}\cdot\vec{n}S = B\ell x$; 1

the induced emf e : $e = -\frac{d\varphi}{dt} = -B\ell\frac{dx}{dt} = -B\ell V$ 1

The voltage across M and N of the rod is then written:

$u_{NM} = -e = B\ell V$. since i goes out from the point M ; so the positive pole of equivalent generator is connected to M. ½

2) a) By applying Newton's second law: $\vec{F} + m\vec{g} + \vec{R}_N = \frac{d\vec{P}}{dt} = m\frac{d\vec{V}}{dt}$; 1

After projection along the direction of motion,

we find: $F = iB\ell = m\frac{dV}{dt}$; 1

$i = C\frac{du_C}{dt} \Rightarrow m\frac{dV}{dt} = B\ell C\frac{du_C}{dt} \Rightarrow mV = B\ell C\, u_C + cte$ 1

At t = 0, V = 0 and $u_C = 0 \Rightarrow cte = 0$, thus: $V = \frac{B\ell C}{m}u_C$. 1

b) By applying the law of addition of voltages, we obtain:

$u_{ND} = u_{NM} + u_{MA} + u_{AD} \Rightarrow E = Ri + B\ell V + u_C$; 1

$$\Rightarrow E = RC\frac{du_C}{dt} + \frac{B^2\ell^2 C}{m}u_C + u_C$$

$$\Rightarrow E = RC\frac{du_c}{dt} + \left(\frac{B^2\ell^2 C + m}{m}\right)u_c$$ 1

3) a) At $t_0 = 0$, $u_C = 0 \Rightarrow a = b$ and $u_c = a - a\, e^{\frac{-t}{\tau}}$; $\dfrac{du_C}{dt} = \dfrac{a}{\tau}\, e^{\frac{-t}{\tau}}$ ½

$$\Rightarrow E = RC\frac{a}{\tau}e^{\frac{-t}{\tau}} + \frac{B^2\ell^2C+m}{m}a - \frac{B^2\ell^2C+m}{m}a\,e^{\frac{-t}{\tau}}$$ ½

$$\Rightarrow a = \frac{mE}{B^2\ell^2C+m} = \frac{0.10\times 6}{1^2\times(0.10)^2\times 0.1+0.10} = 5.94\text{ V}$$ ½

and $\tau = \dfrac{mRC}{B^2\ell^2C+m} = 0.495$ s ½

So: $u_C = 5.94[1 - e^{-2.02t}]$

b) $V = \dfrac{B\ell C}{m}u_C \Rightarrow V = 0.1\times 5.94[1 - e^{-2.02t}]$

$V = 0.594[1 - e^{-2.02t}]$ (in m/s) 1

$i = C\dfrac{du_C}{dt} = 0.1\times 5.94\times 2.02\, e^{-2.02t} = 1.2\, e^{-2.02t}$ (in A) 1

c) $V = \dfrac{dx}{dt} = 0.594[1 - e^{-2.02t}] \Rightarrow x = 0.594\,t + \left(\dfrac{0.594}{2.02}\right)e^{-2.02t} + cte$ 1

At $t_0 = 0$; $0 = 0 + 0.297 + cte$

$\Rightarrow cte = -0.297$ m $\Rightarrow x = 0.594\,t + 0.297[e^{-2.02t} - 1]$. ½

d) i) The steady state is reached for : $t_1 = 5\tau = 5\times 0.495 = 2.475$ s 1

ii) The charge Q of the capacitor: $Q = Cu_C = 0.1\times 5.94 = 0.594$ C 1

The abscissa x_1 of the rod:

$x_1 = 0.594\times 2.475 + 0.297[e^{-5} - 1] = 1.47 - 0.297 = 1.17$ m. ½

Starting from the instant t_1 the motion is uniform, since V becomes constant. ½