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الجامعة اللبنانية
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Entrance Exam 2016 - 2017 **Mathematics** **Duration : 3 hours**
The distribution of grades is over 50 **July 02 , 2016**

I- (7 points) The complex plane is referred to a direct orthonormal system $(O;\ \vec{u}, \vec{v})$.

Let f be the mapping that associates to each point M with affix z, the point M' with affix z' such that $z' = z^2 - 4z$.

1- Let M_1 and M_2 be two distinct points with respective affixes z_1 and z_2.

a) Prove that , if M_1 and M_2 are symmetric with respect to the axis of abscissas then , their images by f, M_1' and M_2', are also symmetric with respect to the axis of abscissas .

b) Prove that, if M_1 and M_2 are symmetric with respect to the point E of affix 2 then, M_1' and M_2' are confounded.

c) Determine the image by f of the point A with affix $z_A = -1+2i$. Deduce the image by f of each of the points B and C with respective affixes $z_B = -1-2i$ and $z_C = 5-2i$.

2- Verify that $z'+4 = (z-2)^2$.

3- Let M be a point , with affix z, belonging to the circle (C) of center E and radius 2.

a) Justify that $z = 2 + 2e^{i\theta}$ where θ is the measure in radians of an oriented angle.

b) As θ traces the interval $[0\ ;\ \frac{\pi}{2}]$, determine and <u>draw</u> the sets (γ) and (γ') of M and M' respectively .

II- (7 points) Consider the function f defined on the interval $]0\ ; +\infty[$ by $f(x) = \frac{\ell n\, x}{\sqrt{x}} - x + 1$.

1- Let g be the function defined on $]0\ ; +\infty[$ by $g(x) = 2 - 2x\sqrt{x} - \ell n\, x$.
Determine the sense of variations of g and calculate $g(1)$. Deduce the sign of $g(x)$.

2- a) Justify that f is differentiable and prove that the sign of $f'(x)$ is that of $g(x)$ in $]0\ ; +\infty[$.

b) Set up the table of variations of f and deduce the sign of $f(x)$ in $]0\ ; +\infty[$.

3- Consider the sequence (U_n) of first term U_0 , $U_0 \in [1\ ;\ 2]$, such that , for all n, $U_{n+1} = 1 + \frac{\ell n(U_n)}{\sqrt{U_n}}$.

a) Prove that, for all x in $[1\ ;\ 2]$, $0 \le \frac{\ell n\, x}{\sqrt{x}} \le 1$.

b) Prove by induction on n that , for all n in IN , $U_n \in [1\ ;\ 2]$.

4- a) Verify that , for all n in IN , $U_{n+1} - U_n = f(U_n)$ and determine the sense of variations of (U_n).

b) Prove that (U_n) is convergent and calculate its limit.

III- (10 points) The complex plane is referred to a direct orthonormal system $(O\,;\,\vec{u}\,,\vec{v}\,)$.

To each point M with affix z, we associate the points N and P of respective affixes z^2 and z^4.

1- Determine the set of values of z so that M, N and P are 3 distinct points.

2- a) When the points M, N and P are distinct, prove that the triangle MNP is right at N if and only if $z^2 + z$ is a pure imaginary number.

b) Prove that the set (γ) of points $M(x\,;\,y)$ such that the triangle MNP is right at N is the hyperbola (H) of equation $\left(x+\dfrac{1}{2}\right)^2 - y^2 = \dfrac{1}{4}$ deprived of two points to be determined.

3- a) Determine the center I, the focus F with positive abscissa and the eccentricity of (H).

b) Draw (H) and precise the set (γ). (***Graph unit : 2 cm***)

4- Let $L(\alpha\,;\,\beta)$ be a variable point of (H) other than its vertices.

a) Write an equation of the tangent (δ) and an equation of the normal (δ') to (H) at L.

b) Determine the points of intersection E and E' of (δ) and (δ') with the focal axis of (H) and prove that $\overline{IE} \times \overline{IE'} = IF^2$.

IV- (5 points) Given an urn containing 6 red balls and 4 blue balls.

A game consists in two parts :

1- In the first part the player draws randomly and simultaneously 3 balls from the urn.
Consider the events R_2 : " The player gets only 2 red balls " and R_3 : " The player gets 3 red balls ".
Prove that $p(R_2) = 0.5$ and calculate $p(R_3)$. Deduce the probability that the player gets at most one red ball.

2- If the player gets at least 2 red balls, he is qualified for the second part that consists in drawing one ball at random from the seven remaining balls in the urn.

a) Calculate the probability that the player gets one red ball in the second part of the game knowing that he got 3 red balls in the first part of the game.

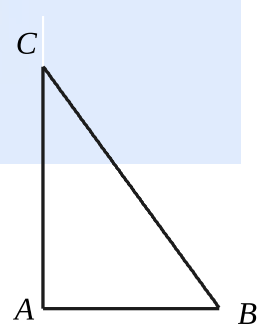
b) Calculate the probability of the event R : " the player gets one red ball in the second part ".

c) Calculate the probability that the player has got 2 red balls in the first part knowing that he got one red ball in the second part.

V- (8 points) Given in an oriented plane, a triangle ABC such that $(\overrightarrow{AB}\,;\,\overrightarrow{AC}) = \dfrac{\pi}{2}\ (2\pi)$,

$AB = 1$ and $AC = \lambda$ where λ is a given real number such that $\lambda > 1$.

Let S be the similitude that transforms B into A and A into C.



1- Determine the ratio and a measure of the angle of S.

2- Let I be the center of S.

a) Determine the nature and the elements of $S \circ S$. Deduce that I belongs to $]BC[$.

b) Prove that I belongs to the circle of diameter $[AB]$ and plot I.

3- Let D be the image of C by S.

a) Calculate CD in terms of λ.

b) Prove that the points A, I and D are collinear .

c) Prove that (CD) and (AB) are parallel. Construct D.

4- Let E be the orthogonal projection of B on (CD) and $F = S(E)$.

Describe the construction of F and determine the nature of the quadrilateral $BFDE$.

VI - (13 points) Consider the function f defined on $]0\,;+\infty[$ by $f(x) = (3 + \ell n\, x)e^{-x}$.

1- Let h be the function defined on $]0\,;+\infty[$ by $h(x) = \dfrac{1}{x} - 3 - \ell n\, x$.

a) Calculate $h'(x)$ and prove that $h'(1) = h(1)$.

b) Set up the table of variations of h.

c) Prove that the equation $h(x) = 0$ has in $]0\,;+\infty[$ a unique solution α and that $\alpha \in]0.45\,;0.46[$.

d) Determine the sign of $h(x)$.

2- Let (C) be the representative curve of the function f in an orthonormal system .

a) Calculate $\lim\limits_{x \to 0^+} f(x)$.

b) Verify that , for all $x > 0$, $f(x) = 3e^{-x} + \dfrac{\ell n\, x}{x} \times \dfrac{x}{e^x}$ and determine $\lim\limits_{x \to +\infty} f(x)$.

c) Determine the function f', the derivative of f, and verify that , for all $x > 0$, $f'(x) = h(x)e^{-x}$.

d) Set up the table of variations of f .

3- a) Prove that $f''(x) = (h'(x) - h(x))e^{-x}$.

b) Prove that the function g defined on $]0\,;+\infty[$ by $g(x) = h'(x) - h(x)$ vanishes once by changing sign.

c) Deduce that (C) has a point of inflection to be determined.

d) Prove that $f(\alpha) = \dfrac{1}{\alpha e^{\alpha}}$ and determine the approximate value of $f(\alpha)$ corresponding to $\alpha = 0.45$.

e) Determine the point of intersection of (C) with the axis of abscissas. Draw (C) **(Graph *unit: 4 cm*).**

Entrance Exam 2016 - 2017 *Solution of Mathematics* *Duration : 3 hours*
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Exercise 1

1- a) If M_1 and M_2 are symmetric with respect to the axis of abscissas $x'x$ then , $z_2 = \overline{z_1}$.
the affixes of the images M_1' and M_2' of M_1 and M_2 are $z_1' = z_1^{\,2} - 4z_1$ and $z_2' = z_2^{\,2} - 4z_2$.
$z_2' = \overline{z_1}^{\,2} - 4\overline{z_1} = \overline{z_1^{\,2} - 4z_1} = \overline{z_1'}$; therefore , M_1' and M_2' are also symmetric with respect to $x'x$.

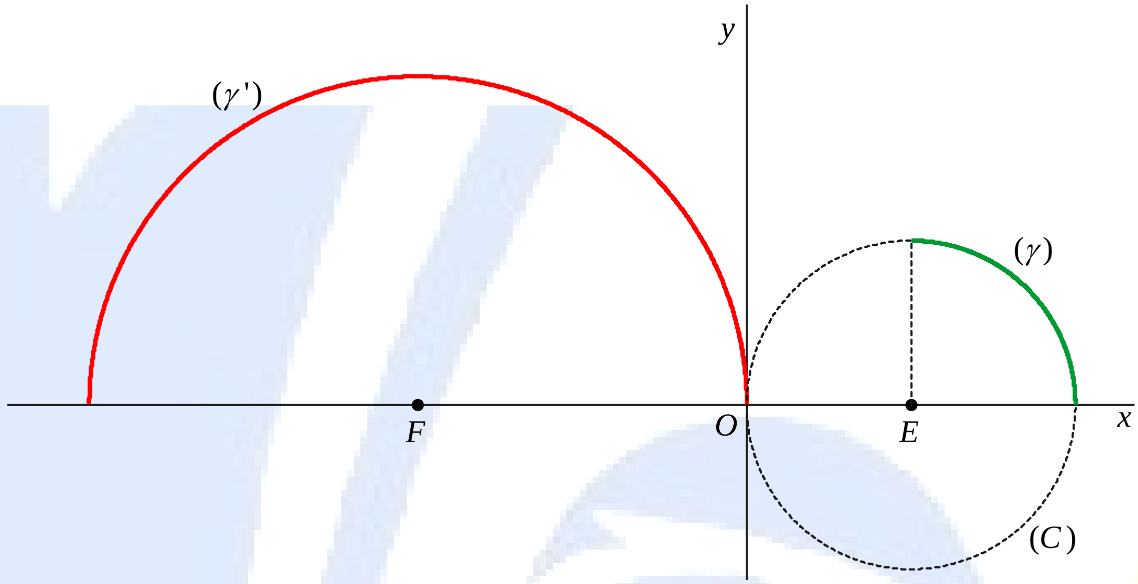
b) If M_1 and M_2 are symmetric with respect to E with affix 2 then , $z_2 = 4 - z_1$.
$z_2' = z_2^{\,2} - 4z_2 = (4 - z_1)^2 - 4(4 - z_1) = z_1^{\,2} - 4z_1 = z_1'$; therefore , M_1' and M_2' are confounded .

c) A is the point with affix $z_A = -1 + 2i$; its image is the point A' with affix $z' = (-1+2i)^2 - 4(-1+2i) = 1 - 12i$.
$z_B = -1 - 2i = \overline{z_A}$ then , B is the symmetric of A with respect to $x'x$; therefore the image of B by f is the point B' symmetric of A' with respect to $x'x$ which is the point of affix $-1-12i$.
$z_C = 5 - 2i$, $z_A + z_C = 4 = 2z_E$ then , C is the symmetric of A with respect to E ; therefore $C' = A'$.

2- $z' + 4 = z^2 - 4z + 4 = (z-2)^2$.

3- a) If M belongs to (C) then $EM = 2$; therefore $|z - 2| = 2$.
If θ is an argument of $z - 2$ then , $z - 2 = 2e^{i\theta}$; that is $z = 2 + 2e^{i\theta}$.

b) As θ traces the interval $[0 ; \frac{\pi}{2}]$, the set (γ) of M is the quarter of circle (C) corresponding to $x \geq 2$ and $y \geq 0$.
$z' + 4 = (z-2)^2$ then $|z' + 4| = 4$ and $\arg(z'+4) = 2\arg(z+2) = 2\theta$.
If F is the point with affix 4 then , $FM' = 4$ and $(\vec{u} \,;\, \overrightarrow{FM'}) = 2\theta$.
As θ traces the interval $[0 ; \frac{\pi}{2}]$, 2θ traces the interval $[0 ; \pi]$; therefore , the set (γ') of M' is the semi circle of center F and radius 4 lying above the axis of abscissas .
Drawing (γ) and (γ') .



Exercise 2

1- The function g is defined on $]0\,;+\infty[$ by $g(x)=2-2x\sqrt{x}-\ell n\,x$.

$g'(x)=-3\sqrt{x}-\dfrac{1}{x}$; for all x in $]0\,;+\infty[$, $g'(x)<0$ then, g is strictly decreasing ; $g(1)=0$.

For all x in $]0\,;1[$, $g(x)>g(1)$; that is, $g(x)>0$;

For all x in $]1\,;+\infty[$, $g(x)<g(1)$; that is, $g(x)<0$.

2- a) Each of the functions $x\to x\sqrt{x}$ and $x\to \ell n\,x$ is differentiable on $]0\,;+\infty[$ then, f is differentiable .

$f'(x)=\dfrac{2-\ell n\,x-2x\sqrt{x}}{2x\sqrt{x}}=\dfrac{g(x)}{2x\sqrt{x}}$ then, the sign of $f'(x)$ is that of $g(x)$ in $]0\,;+\infty[$.

b) $\lim\limits_{x\to 0^+} f(x)=-\infty$.

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\left(2\dfrac{\ell n\sqrt{x}}{\sqrt{x}}-x+1\right)=-\infty$.

Table of variations of f

x	0		1		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	$-\infty$	$\nearrow$	0	$\searrow$	$-\infty$

The function f has an absolute maximum equals 0 then, for all x in $]0\,;+\infty[-\{1\}$, $f(x)<0$.

3- a) For all x in $[1\,,2]$, $1\le\sqrt{x}\le\sqrt{2}$ and $0\le \ell n\,x\le 1$ then $0\le\dfrac{1}{\sqrt{x}}\le 1$ and $0\le \ell n\,x\le 1$;

therefore $0 \le \frac{\ell n\, x}{\sqrt{x}} \le 1$.

b) ▪ $U_0 \in [1\,;\,2]$.

▪ If , for a certain n , $U_n \in [1\,;\,2]$ then , $0 \le \frac{\ell n(U_n)}{\sqrt{U_n}} \le 1$; $1 \le 1 + \frac{\ell n(U_n)}{\sqrt{U_n}} \le 2$; that is $1 \le U_{n+1} \le 2$.

Therefore , for all n in IN , $U_n \in [1\,,\,2]$.

4- a) For all n in IN , $U_{n+1} - U_n = 1 - U_n + \frac{\ell n(U_n)}{\sqrt{U_n}} = f(U_n)$ and for all x in $[1\,;\,2]$, $f(x) \le 0$ then , for all n in IN , $U_{n+1} - U_n \le 0$ and (U_n) is a decreasing sequence .

b) (U_n) is decreasing and bounded then , it converges to a limit $\ell \in [1\,,\,2]$ such that $\ell = 1 + \frac{\ell n\, \ell}{\sqrt{\ell}}$.

Therefore $\frac{\ell n\, \ell}{\sqrt{\ell}} - \ell + 1 = 0$; $f(\ell) = 0$; $\ell = 1$.

Exercise 3

1- ▪ $M = N$ if and only if $z^2 = z$; that is $z = 0$ or $z = 1$.

▪ $N = P$ if and only if $z^4 = z^2$; that is $z^2 = 0$ or $z^2 = 1$; $z = 0$ or $z = 1$ or $z = -1$.

▪ $M = P$ if and only if $z^4 = z$; that is $z = 0$ or $z^3 = 1$; $z = 0$ or $z = 1$ or $z = j$ or $z = \bar{j}$.

Finally , the points M , N and P are distinct in pairs if and only if z a complex not belonging to the set $S = \left\{0\,;\,1\,;\,-1\,;\,j\,;\,\bar{j}\right\}$ where $j = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

2- a) If the points M , N and P are distinct in pairs , the triangle MNP is right at N if and only if $\frac{z^4 - z^2}{z - z^2}$ is a pure imaginary number ; that is $-\frac{(z^2 - z)(z^2 + z)}{z^2 - z}$ is a pure imaginary number ; $z^2 + z$ is a pure imaginary number .

b) The triangle MNP is right at N if and only if $z \notin S$ and $z^2 + z$ is a pure imaginary number .

$z^2 + z = x^2 - y^2 + x + (2x+1)yi$

When $z \notin S$, $z^2 + z \ne 0$ then , $z^2 + z$ is a pure imaginary number if and only if

$x^2 - y^2 + x = 0$; $\left(x + \frac{1}{2}\right)^2 - y^2 = \frac{1}{4}$.

The set (γ) of points $M(x\,;\,y)$ is the hyperbola (H) of equation $\left(x+\frac{1}{2}\right)^2 - y^2 = \frac{1}{4}$ deprived from the points $O(0\,;\,0)$ and $A(-1\,;\,0)$, the points of (H) whose affixes belong to S.

3- a) For the hyperbola (H) :

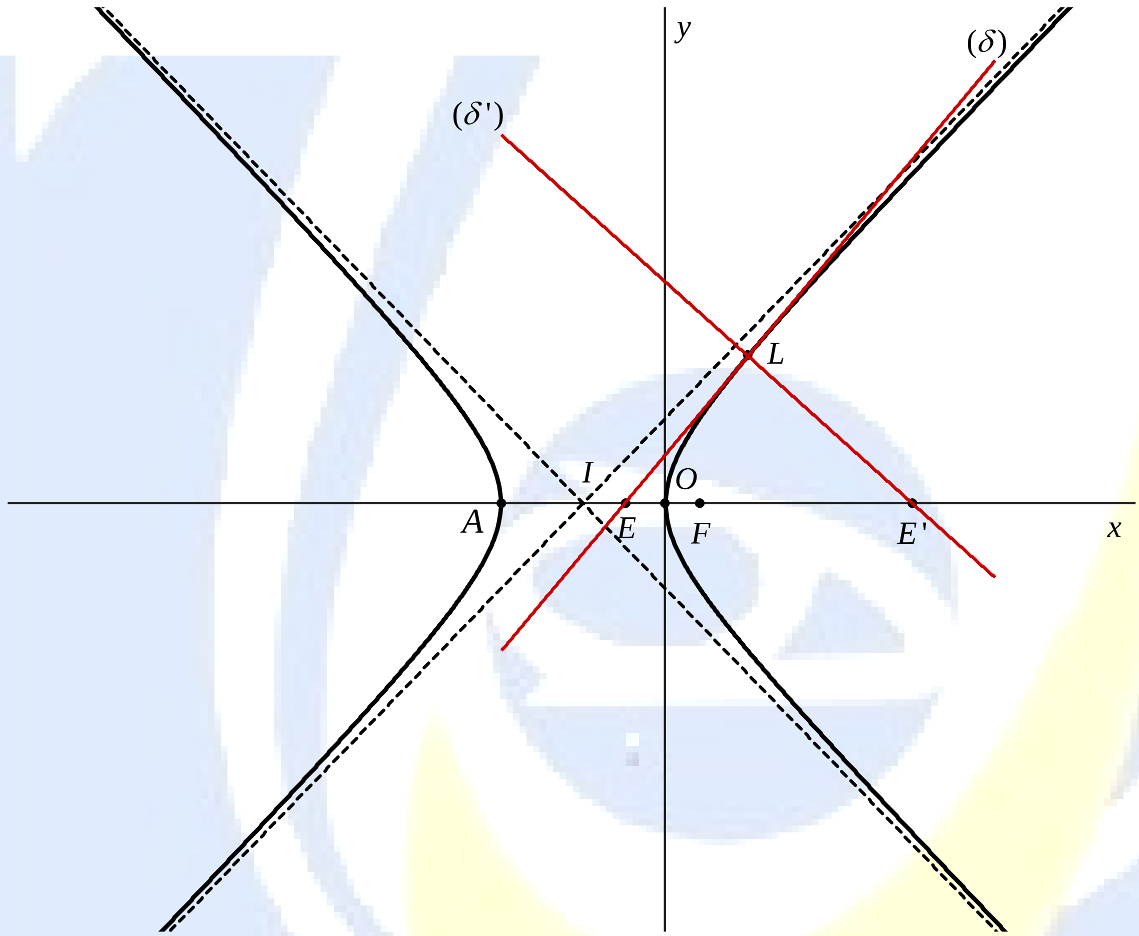
- The center is $I(-\frac{1}{2}\,;\,0)$;
- $a^2 = b^2 = \frac{1}{4}$ then (H) is a rectangular hyperbola with eccentricity is $e = \sqrt{2}$
- The focal axis is $x'x$ and $c = a\sqrt{2} = \frac{\sqrt{2}}{2}$ then , the focus with positive abscissa is $F(-\frac{1}{2}+\frac{\sqrt{2}}{2}\,;\,0)$.

b) The vertices of (H) are $O(0\,;\,0)$ and $A(-1\,;\,0)$.

The asymptotes are the straight lines of equations $y = -x - \frac{1}{2}$ and $y = x + \frac{1}{2}$

Drawing (H). (***Graph unit : 2 cm***)

The set (γ) is the hyperbola (H) deprived from its vertices .



4- Let $L(\alpha\ ;\ \beta)$ be a variable point of (H) other than its vertices .

a) An equation of the tangent (δ) to (H) at L is $\left(\alpha+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)-\beta\, y=\frac{1}{4}$;

An equation of the normal (δ') to (H) at L is $\frac{1}{4}\times\frac{x+\frac{1}{2}}{\alpha+\frac{1}{2}}+\frac{1}{4}\times\frac{y}{\beta}=c^2=\frac{1}{2}$.

b) (δ) and (δ') cut the focal axis $x'x$ at $E(\frac{1}{4\alpha+2}-\frac{1}{2}\,;0)$ and $E'(2\alpha+\frac{1}{2}\,;0)$.

$\overline{IE}=\frac{1}{4\alpha+2}$, $\overline{IE}'=2\alpha+1$ then , $\overline{IE}\times\overline{IE}'=\frac{1}{2}=IF^2$.

Exercise 4

The urn contains 10 balls then , there are ${}_{10}C_3$ equiprobable ways of selecting 3 balls from the urn .

1- There are 6 red and 4 blue balls in the urn then

$p(R_2)=\frac{{}_6C_2\times{}_4C_1}{{}_{10}C_3}=\frac{60}{120}=0.5$ and $p(R_3)=\frac{{}_6C_3}{{}_{10}C_3}=\frac{20}{120}=\frac{1}{6}$.

Let L be the event : " The player gets at most one red ball " .

$L=\overline{R_3\cup R_2}$ where R_2 and R_3 are incompatible ; therefore $p(L)=1-p(R_2)-p(R_3)=\frac{1}{3}$.

2- a) If 3 red balls are drawn in the first part of the game then , for the second part , the urn contains 3 red and 4 blue balls ; the required probability is $p_1=\frac{3}{7}$.

b) If 2 red balls are extracted in the first part of the game then , for the second part , the urn contains 4 red and 3 blue balls and $p_2=p(\textit{selecting one red ball from this urn})=\frac{4}{7}$.

Therefore , $p(R)=p(R_3)\times p_1+p(R_2)\times p_2=\frac{1}{6}\times\frac{3}{7}+\frac{1}{2}\times\frac{4}{7}=\frac{5}{14}$.

c) The required probability is $p(R_2/R)=\frac{p(R_2\cap R)}{p(R)}=\frac{1}{2}\times\frac{4}{7}\div\frac{5}{14}=\frac{4}{5}$.

Exercise 5

1- $S(B)=A$ and $S(A)=C$ then , the ratio of S is $\dfrac{AC}{AB}=\lambda$ and its angle is $(\overrightarrow{BA}\ ;\ \overrightarrow{AC})=-\dfrac{\pi}{2}\quad(2\pi)$

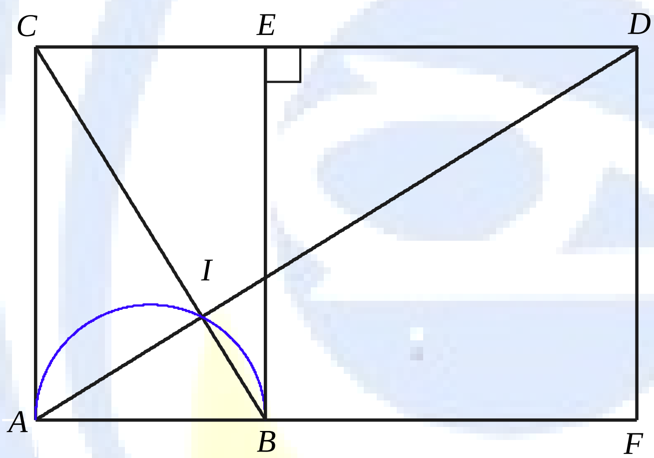
2- a) $S=S(I\ ;\lambda\ ;-\dfrac{\pi}{2})$ then , $S\circ S$ is the similitude of center I , ratio λ^2 and angle $-\pi$.

Therefore $S\circ S$ is the negative dilation of center I and ratio $-\lambda^2$.

$S\circ S(B)=S\big(S(B)\big)=S(A)=C$ then , $\overrightarrow{IC}=-\lambda^2\,\overrightarrow{IB}$; therefore , I belongs to $]BC[$.

b) $S(B)=A$ then , $(\overrightarrow{IB}\ ;\ \overrightarrow{IA})=-\dfrac{\pi}{2}\quad(2\pi)$; therefore I belongs to the circle (γ) of diameter $[AB]$.

I is the point of intersection of the segment $]BC[$ and circle (γ) .



3- a) $S(C)=D$ and $S(A)=C$ then , $CD=\lambda\,AC=\lambda^2$.

b) $S\circ S(A)=S\big(S(A)\big)=S(C)=D$ where $S\circ S$ is a dilation of center I then , A , I and D are collinear .

c) $S\circ S(A)=D$ and $S\circ S(B)=C$ where $S\circ S$ is a dilation then , (CD) and (AB) are parallel .

D is the point of intersection of (AI) and the parallel to (AB) passing through C .

4- E is the orthogonal projection of B on (CD) .

An angle of S is $-\dfrac{\pi}{2}$ then , any straight line and its image by S are perpendicular .

$S(C)=D$ then , $S\big((CE)\big)$ is the perpendicular to (CE) at D which is the perpendicular to (AB) passing through D .

$S(B)=A$ then , $S\big((BE)\big)$ is the perpendicular to (BE) passing though A which is (AB) .

Therefore F is the orthogonal projection of D on (AB) .

$BFDE$ is a rectangle for having 4 right angles .

Exercise 6

1- The function h is defined on $]0\,;+\infty[$ by $h(x)=\dfrac{1}{x}-3-\ell n\,x$.

a) $h'(x)=\dfrac{-1}{x^2}-\dfrac{1}{x}$; $h'(1)=h(1)=-2$

b) $\lim\limits_{x\to 0^+} h(x)=+\infty$ and $\lim\limits_{x\to +\infty} h(x)=-\infty$.

Table of variations of h.

x	0		$+\infty$
$h'(x)$		$-$	
$h(x)$	$+\infty$	$\searrow$	$-\infty$

c) The function h is continuous and strictly decreasing on $]0\,;+\infty[$ and 0 belongs to $h(]0\,;+\infty[)$ which is IR then , the equation $h(x)=0$ has a unique solution α in $]0\,;+\infty[$.

$h(0.45)\approx 0.02>h(\alpha)=0>h(0.46)\approx -0.05$ and h is strictly decreasing then , $\alpha\in]0.45\,;0.46[$.

d) h is strictly decreasing then :

for all $x<\alpha$, $h(x)>h(\alpha)$; $h(x)>0$ and for all $x>\alpha$, $h(x)<h(\alpha)$; $h(x)<0$.

2- a) $\lim\limits_{x\to 0^+} f(x)=-\infty$.

b) For all $x>0$, $f(x)=3e^{-x}+e^{-x}\ell n\,x=3e^{-x}+\dfrac{\ell n\,x}{x}\times\dfrac{x}{e^x}$.

$\lim\limits_{x\to +\infty} e^{-x}=0$, $\lim\limits_{x\to +\infty}\dfrac{\ell n\,x}{x}=0$ and $\lim\limits_{x\to +\infty}\dfrac{x}{e^x}=0$ then , $\lim\limits_{x\to +\infty} f(x)=0$.

c) $f'(x)=\dfrac{1}{x}e^{-x}-(3+\ell n\,x)e^{-x}=h(x)e^{-x}$.

d) The sign of $f'(x)$ is that of $h(x)$.

Table of variations of f .

x	0		α		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	$-\infty$	$\nearrow$	$f(\alpha)$	$\searrow$	0

3- a) $f'(x)=h(x)e^{-x}$ then , $f''(x)=h'(x)e^{-x}-h(x)e^{-x}$

$f''(x)=\big(h'(x)-h(x)\big)e^{-x}$.

b) $g(x)=h'(x)-h(x)=\dfrac{-1}{x^2}-\dfrac{2}{x}+3+\ell n\,x$.

$g'(x)=\dfrac{2}{x^3}+\dfrac{2}{x^2}+\dfrac{1}{x}$; for all $x>0$, $g'(x)>0$ then , g is strictly increasing .

The function g is continuous , strictly increasing and $g(1)=h'(1)-h(1)=0$ then , $g(x)$ changes sign at 1 from negative to positive .

c) $f''(x) = g(x)e^{-x}$ then , the sign of $f''(x)$ which is that of $g(x)$, changes also at 1 ; therefore , the point $I(1\,;\,3e^{-1})$ is the point of inflection of (C) .

d) $f(\alpha) = (3 + \ell n\alpha)e^{-\alpha}$ with $\ell n\alpha = \dfrac{1}{\alpha} - 3$ then , $f(\alpha) = \dfrac{1}{\alpha e^{\alpha}}$.

For $\alpha = 0.45$, $f(\alpha) \approx 1.42$.

e) $f(x) = 0$ is equivalent to $\ell n x = -3$; $x = e^{-3}$.

(C) cuts the axis of abscissas at the point $(e^{-3}\,;\,0)$.

$\lim\limits_{x \to 0^{+}} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = 0$ then , the asymptotes of (C) are the axes of coordinates .

Drawing (C) in an orthonormal system . (***Unit : 4 cm***)