LEBANESE UNIVERSITY

FACULTY OF ENGINEERING

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Entrance Exam 2011 - 2012 **Mathematics** **Duration : 3 hours**

02, July 2011

The distribution of grades is over 25

I- (3 points) The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

Let $z = re^{i\alpha}$ where r is a positive real number such that $r \neq 1$.

Consider the points A, B, C and D of respective affixes $z_A = z$, $z_B = \frac{1}{z}$, $z_C = \frac{\bar{z}}{z^2}$ and $z_D = -\bar{z}$.

1- Determine the exponential form of $\frac{z_A}{z_C}$. Deduce the set of values of α such that O belongs to the segment $]AC[$.

2- Suppose in this part that $\alpha = \frac{\pi}{4}$.

a) Prove that $z_C - z_D = \overline{z_A - z_B}$

b) Calculate $z_A - z_D$ and $z_B - z_C$ in terms of r and prove that these numbers are 2 distinct real numbers.

c) Prove that $ABCD$ is an isosceles trapezoid whose diagonals intersect at O.

II- (2.5 points) Consider the sequence (U_n) of first term U_0 such that , for all n, $3U_{n+1} - 6 = (U_n - 2)(U_n + 1)$.

1- If the sequence (U_n) converges what is the value of its limit ℓ ?

2- Prove that if $U_0 \in \{-1;2\}$ then, for all $n \geq 1$, $U_n = 2$.

3- Calculate $3U_{n+1} - 3U_n$ in terms of U_n and prove that if $U_0 \notin \{-1;2\}$ then (U_n) is increasing.

4- Prove that if $U_0 \in]-1;2[$ then, for all natural number n, $U_n \in]-1;2[$ and (U_n) is convergent .

5- Prove that if $U_0 > 2$ then, for all natural number n, $U_n > 2$ and (U_n) is divergent .

III- (4 points) The plane is referred to the direct orthonormal system $(O;\vec{u};\vec{v})$.

Let T be the transformation whose complex relation is $Z = (3+4i)\bar{z} - 8 - 4i$.

1- Prove that T has an invariant point whose coordinates are to be determined.

2- Determine the complex relation of the dilation h of center $\omega(2;1)$ and ratio 5.

3- Let $S = T \circ h^{-1}$.

a) Prove that $z' = (\frac{3}{5} + \frac{4}{5}i)\bar{z}$ is the complex relation of S.

b) Determine the set (d) of invariant points of S and verify that ω and O belong to (d).

4- Let $M(z)$ be any point of the plane and $M'(z')$ its image by S.

Prove that $|z'| = |z|$ and $|z' - z_\omega| = |z - z_\omega|$. Deduce that S is the reflection of axis (d).

5- a) Prove that $T = S \circ h$.

b) A point M not belonging to (d) being given. Describe the construction of the point $M' = T(M)$.

IV- (2.5 points) A statistical study concerning a certain illness is done on families having 2 children : one girl and one boy . We found the following results :

- $50\,\%$ of boys and $20\,\%$ of girls are attacked by the illness .
- In the families where the boy is attacked , the girl is also attacked in $25\,\%$ of the cases .

One of the families under study is selected at random .
Calculate the probability of each of the following events :

A : " the two children are attacked by the illness " ;
B : " only one of the two children is attacked by the illness " ;
C : " no one of the two children is attacked by the illness " ;
D : " the boy is attacked knowing that the girl is " ;
E : " the girl is attacked knowing that the boy is not " .

V- (7 points) The plane is referred to a direct orthonormal system $(O\,;\,\vec{i}\,,\,\vec{j}\,)$.

A- Consider the differential equation $(E)\,:\; y'-y=e^{x}-1\;\;;\;\; x\in IR$.

Let z be a differentiable function such that $y=z\,e^{x}+1$.

1- Determine the differential equation (1) whose general solution is the function z .

2- Solve the equation (1) and deduce the general solution of (E) .

B- Consider the function p defined on IR by $p(x)=(x+a)e^{x}+1$ where a is a real parameter.
Let (γ) be the representative curve of p .

1- Prove that , for all real numbers a , (γ) has a fixed asymptote to be determined .

2- a) Prove that the solutions of the equations $p''(x)=0$; $p'(x)=0$; $p(x)=1$ are 3 consecutive terms of an increasing arithmetic sequence whose common difference is to be determined .

b) Determine a so that the fourth term of this sequence is the solution of the equation $p(x)=e+1$.

3- a) Set up the table of variations of p and prove that , for all a in IR , p has a minimum .

b) Determine , as a varies , the set of the point S of (γ) corresponding to the minimum of p .

c) Determine the set of values of a so that , for all x in IR , $p(x)\geq 0$.

d) Deduce the sign of the functions f and g defined on IR by $f(x)=x\,e^{x}+1$ and $g(x)=(x-1)e^{x}+1$.

C- Consider the function h such that $h(x)=\dfrac{xe^{x}}{x\,e^{x}+1}$. Let (L) be the representative curve of h .

1- a) Justify that h is defined on IR .

b) Set up the table of variations of h .

2- a) Verify that (L) passes through O and write an equation of the tangent (d) to (L) at this point .

b) Verify that , for all x in IR , $h(x)-x=-\left[\dfrac{g(x)}{f(x)}\right]x$.

c) Determine the relative position of (L) and (d) . What does the point O represent for (L) ? Draw (L) and (d)

3- a) Prove that the restriction of h to the interval $[-1\,;+\infty[$ has an inverse function h^{-1}.

b) Prove that the representative curve (L') of h^{-1} is tangent to (L) at O. Draw (L').

VI- (6 points) The plane is referred to a direct orthonormal system $(O\,;\vec{i},\vec{j})$.

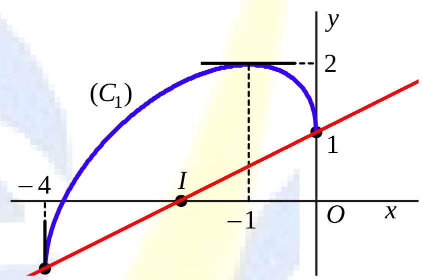
A- Consider the straight lines (δ) and (Δ) of respective equations $x=-4$ and $x-2y+2=0$.
Let M be any point lying between (δ) and the axis of ordinates $y'y$. Designate by H, H' and K the orthogonal projections of M on $y'y$, (δ) and (Δ) respectively. Prove that the set of points M such that $5MK^2=3MH\times MH'$ is the curve (C) of equation $(x-2y+2)^2=-3x(x+4)$.

B- Consider the curve (C_1) of equation $y=\dfrac{1}{2}\left(x+2+\sqrt{-3x^2-12x}\right)$.

1- Determine an equation of the curve (C_2), the symmetric of (C_1) with respect to the point $I(-2\,;0)$.

2- Prove that $(C)=(C_1)\cup(C_2)$.

3- The curve (C_1) is drawn in the adjacent figure. Draw (C). (***Unit :*** $2\,\boldsymbol{cm}$)



C- Let r be the rotation of center O and angle $-\dfrac{\pi}{4}$.

1- Prove that $x^2+3y^2+2\sqrt{2}\,x-6\sqrt{2}\,y+2=0$ is an equation of the image (E) of (C) by r.

2- a) Prove that (E) is an ellipse whose area is $S=8\sqrt{3}\,\pi\ cm^2$.

b) Deduce that (C) is an ellipse of center I and calculate its area. Deduce $\displaystyle\int_{-4}^{0}\sqrt{-3x^2-12x}\,dx$.

c) Determine the focal axis of (C) and the coordinates of one of its foci.

Entrance exam 2011-2012 **Solution of Mathematic** **Time: 3 hours**
02, July 2011

Exercise 1

1- ▪ $\frac{z_A}{z_C} = \frac{z^3}{\bar{z}} = \frac{r^3 e^{3i\alpha}}{r e^{-i\alpha}} = r^2 e^{4i\alpha}$.

▪ $(\overrightarrow{OC}\ ;\overrightarrow{OA}) = \arg\left(\frac{z_A}{z_C}\right) = 4\alpha \quad (2\pi)$.

O, A and C are such that $O \in [AC]$ if and only if $(\overrightarrow{OC}\ ;\overrightarrow{OA}) = \pi + 2k\pi$; that is $4\alpha = \pi + 2k\pi$ and $\alpha = \frac{\pi}{4} + k\frac{\pi}{2}$ where $k \in Z$.

2- Suppose in this part that $\alpha = \frac{\pi}{4}$.

a) ▪ $z_C - z_D = \frac{1}{r}e^{-i\frac{3\pi}{4}} + r e^{-i\frac{\pi}{4}} = r e^{-i\frac{\pi}{4}} - \frac{1}{r}e^{i\frac{\pi}{4}} = \overline{z_A - z_B}$.

b) ▪ $z_A - z_D = z + \bar{z} = 2\,\mathrm{Re}(z) = 2r\cos\frac{\pi}{4} = r\sqrt{2}$.

▪ $z_B - z_C = \frac{1}{r}e^{-i\frac{\pi}{4}} - \frac{1}{r}e^{-i\frac{3\pi}{4}} = \frac{1}{r}e^{-i\frac{\pi}{4}} + \frac{1}{r}e^{i\frac{\pi}{4}} = \frac{1}{r}(e^{-i\frac{\pi}{4}} + e^{i\frac{\pi}{4}}) = \frac{1}{r}(2\cos\frac{\pi}{4}) = \frac{\sqrt{2}}{r}$.

▪ For all values of r in $]0\,;+\infty[-\{1\}$, the two numbers $z_A - z_D$ and $z_B - z_C$ are real numbers .

▪ Since $r \neq 1$ then $r \neq \frac{1}{r}$ therefore $z_A - z_D \neq z_B - z_C$.

c) ▪ Each of $z_A - z_D$ and $z_C - z_D$ is a real number the two straight lines (AD) and (BC) are parallel to the axis of abscissas $x'x$, then (AD) and (BC) are parallel .

▪ $AD \neq BC$ since $z_A - z_D \neq z_B - z_C$.

▪ $z_C - z_D = \overline{z_A - z_B}$ then $\left|z_A - z_B\right| = \left|z_C - z_D\right|$ therefore $AB = CD$.

Therefore $ABDC$ is an isosceles trapezoid .

▪ $\frac{z_A}{z_C} = r^2 e^{i\pi} = -r^2$ and $\frac{z_B}{z_C} = -\frac{1}{r^2}$.

Each of $\frac{z_A}{z_C}$ and $\frac{z_B}{z_C}$ is a negative real number then $O \in [AC]$ and $O \in [BD]$ and the diagonals $[AD]$ and $[BC]$ intersect at O .

Exercise 2

The sequence (U_n) is defined by its first term U_0 and by the relation $3U_{n+1} - 6 = (U_n - 2)(U_n + 1)$.

1- If the sequence (U_n) converges then its limit ℓ is such that $3\ell - 6 = \ell^2 - \ell - 2$; that is $\ell^2 - 4\ell + 4 = 0$; $(\ell - 2)^2 = 0$; therefore $\ell = 2$.

2- ▪ $3U_1 - 6 = (U_0 - 2)(U_0 + 1)$; therefore , if $U_0 = 2$ or $U_0 = -1$ then $U_1 = 2$.

▪ If , for a certain value of $n \geq 1$, $U_n = 2$ then , $3U_{n+1} - 6 = (U_n - 2)(U_n + 1) = 0$; that is $U_{n+1} = 2$.

Therefore for all $n \geq 1$, $U_n = 2$.

3- ▪ $3U_{n+1} - 3U_n = U_n^2 - 4U_n + 4 = (U_n - 2)^2$.

▪ For all n in IN $U_{n+1} - U_n \geq 0$ therefore , (U_n) is increasing .

4- ▪ $U_0 \in \left]-1 ; 2\right[$

▪ If , for a certain value of n , $U_n \in \left]-1 ; 2\right[$ then , $U_n + 1 > 0$ and $U_n - 2 < 0$; therefore $3U_{n+1} - 6 < 0$ and $U_{n+1} < 2$.

Therefore , for all n in IN , $U_n < 2$.

On the other hand , the sequence (U_n) is increasing , then , for all n in IN , $U_n \geq U_0 > -1$.

Finally , for all n in IN , $U_n \in \left]-1 ; 2\right[$.

The sequence (U_n) is strictly increasing and bounded from above by 2 ; therefore it is convergent and its limit , according to part 1 , is equal to 2 .

5- ▪ $U_0 > 2$.

▪ If , for a certain value of n in IN , $U_n > 2$ then , $U_n + 1 > 0$ and $U_n - 2 > 0$; therefore $4U_{n+1} - 8 > 0$ and $U_{n+1} > 2$.

Therefore, for all n in IN , $U_n > 2$.

Therefore the sequence (U_n) cannot converge to the eventual limit 2 ; then it is a divergent sequence .

Exercise 3

1- The affix of the invariant $(x\,;\,y)$ of T is the solution of the equation $z=(3+4i)\bar{z}-8-4i$ which is equivalent to $x+iy=(3+4i)(x-iy)-8-4i$; $2(x+2y-4)+4(x-y-1)i=0$; that is $x+2y-4=0$ and $x-y-1=0$; $x=2$ and $y=1$. Finally , the invariant point of T is $\omega(2\,;1)$

2- The complex relation of the dilation $h=h(\omega\,;5)$ is $z'=5z+(1-5)z_{\omega}$; that is $z'=5z-8-4i$.

3- h^{-1} is the dilation of center $\omega(2\,;1)$ and ratio $\frac{1}{5}$; its complex relation is $z'=\frac{1}{5}z+\frac{8}{5}+\frac{4}{5}i$.

a) Let M be any point with affix z .

$$\begin{tikzcd} M(z) \arrow[r, "h^{-1}"] \arrow[rr, bend right, "S\circ h^{-1}"'] & M_1(z_1) \arrow[r, "S"] & M'(z') \end{tikzcd}$$

Figure 13

We have $z_1=\frac{1}{5}z+\frac{8}{5}+\frac{4}{5}i$ and $z'=(3+4i)\overline{z_1}-8-4i$ then

$z'=(3+4i)\,(\frac{1}{5}\bar{z}+\frac{8}{5}-\frac{4}{5}i)-8-4i=(\frac{3}{5}+\frac{4}{5}i)\bar{z}+8+4i-8-4i$; $z'=(\frac{3}{5}+\frac{4}{5}i)\bar{z}$.

b) The set of invariant points of S is the set (d) of points $M(x\,;\,y)$ such that $z=(\frac{3}{5}+\frac{4}{5}i)\bar{z}$;

$x+iy=(\frac{3}{5}+\frac{4}{5}i)(x-iy)$; $2(x-2y)-4(x-2y)i=0$; that is $x-2y=0$.

Therefore (d) is the straight line of equation $x-2y=0$ which passes through ω and O.

4- $M(z)$ is any point of (P) and $M'(z')$ its image by S then $z'=(\frac{3}{5}+\frac{4}{5}i)\bar{z}$.

$$|z'|=\left|(\frac{3}{5}+\frac{4}{5}i)\bar{z}\right|=\left|\frac{3}{5}+\frac{4}{5}i\right|\times|\bar{z}|=|\bar{z}|=|z|\ .$$

$$|z'-z_{\omega}|=\left|(\frac{3}{5}+\frac{4}{5}i)\bar{z}-(\frac{3}{5}+\frac{4}{5}i)\bar{z}_{\omega}\right|=\left|\frac{3}{5}+\frac{4}{5}i\right|\times|\bar{z}-\bar{z}_{\omega}|=|z-z_{\omega}|\ .$$

$|z'|=|z|$ is equivalent to $OM=OM'$ and $|z'-z_{\omega}|=|z-z_{\omega}|$ is equivalent to $\omega M=\omega M'$ then the straight line $(O\omega)$, which is (d), is the perpendicular bisector of $[MM']$.

Therefore S is the reflection of axis (d).

5- a) $S = T \circ h^{-1}$ is equivalent to $S \circ h = (T \circ h^{-1}) \circ h = T \circ (h^{-1} \circ h) = T$.

b) M is a point not belonging to (d) ; $M' = T(M) = (S \circ h)(M) = S(h(M)) = S(N)$ where $N = h(M)$.

Therefore M' is the symmetric with respect to (d) of the image of M under h.

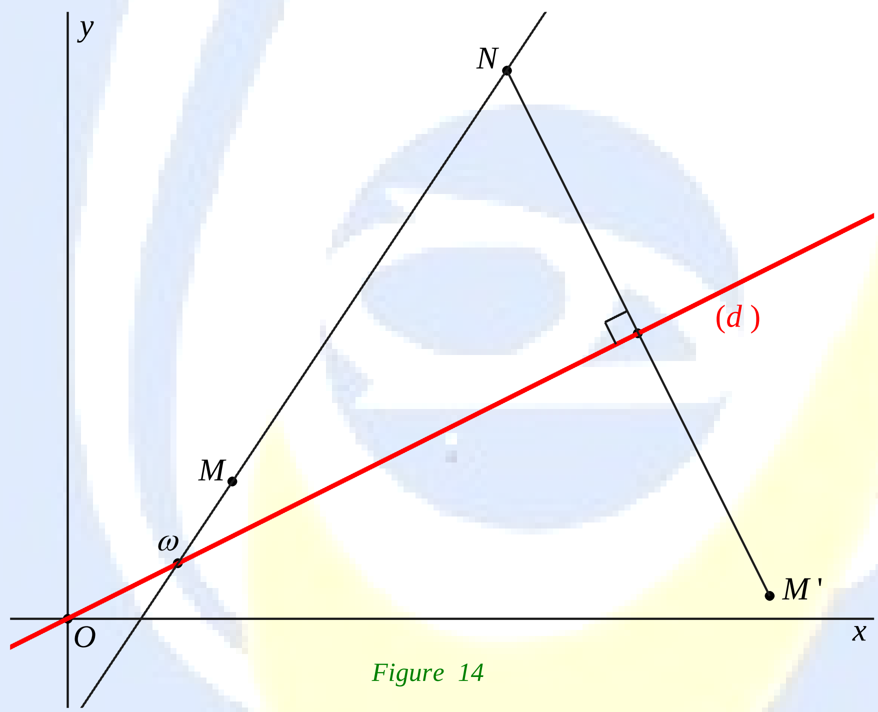


Figure 14

Exercise 4

Consider the events :

M : " the boy of the family is attacked by the illness "

F : " the girl of the family is attacked by the illness ".

- 50% of boys are attacked by the illness then $p(M) = \frac{1}{2}$
- 20% of girls are attacked by the illness then $p(F) = \frac{1}{5}$

▪ In the families where the boy is attacked , the girl is also attacked in 25% of the cases then $p(F/M)=\frac{1}{4}$.

A : " the two children are attacked by the illness " ; $A=M\cap F$.

$p(A)=p(M\cap F)=p(M)\times p(F/M)=\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$

B : " only one of the two children is attacked by the illness " ; $B=M\cup F-M\cap F$.

$p(B)=p(M)+p(F)-2\,p(M\cap F)=p(M)+p(F)-2p(A)=\frac{1}{2}+\frac{1}{5}-\frac{1}{4}=\frac{9}{20}$.

C : " no one of the two children is attacked by the illness " ; $C=\overline{M}\cap\overline{F}=\overline{M\cup F}$.

$p(C)=p(\overline{M\cup F})=1-p(M\cup F)=1-p(M)-p(F)+p(A)=1-\frac{1}{2}-\frac{1}{5}+\frac{1}{8}=\frac{17}{40}$.

D : " the boy is attacked knowing that the girl is " ; $D=M/F$.

$p(D)=p(M/F)=\frac{p(M\cap F)}{p(F)}=\frac{p(A)}{p(F)}=\frac{1}{8}\times 5=\frac{5}{8}$.

E : " the girl is attacked knowing that the boy is not " . $E=F/\overline{M}$.

$p(E)=p(F/\overline{M})=\frac{p(\overline{M}\cap F)}{p(\overline{M})}=\frac{p(F)-p(M\cap F)}{1-p(M)}=\frac{p(F)-p(A)}{1-p(M)}=\left(\frac{1}{5}-\frac{1}{8}\right)\div\left(1-\frac{1}{2}\right)=\frac{3}{20}$.

Exercise 5

A- (E) : $y'-y=e^x-1$; $x\in IR$.

1- If $y=ze^x+1$ then $y'=z'e^x+ze^x$.

By substitution in the equation (E) we find $(z'e^x+ze^x)-(ze^x+1)=e^x-1$; $z'e^x-1=e^x-1$;
$z'e^x=e^x$; $z'=1$; then (1) : $z'=1$.

2- The general solution of the equation (1) is $z=x+a$ where $a\in IR$; therefore the general solution of the equation (E) is $p(x)=(x+a)e^x+1$.

B- The function p is defined on IR by $p(x)=(x+a)e^x+1$ where a is a real parameter .

1- $\lim\limits_{x\to-\infty} xe^x=0$; therefore , for all a in IR , $\lim\limits_{x\to-\infty} p(x)=\lim\limits_{x\to-\infty}(xe^x+ae^x+1)=1$.

Therefore , as a varies , (γ) has a fixed asymptote of equation $y=1$.

2- $p(x)=(x+a)e^x+1$; $p'(x)=(x+a+1)e^x$; $p''(x)=(x+a+2)e^x$

a) ▪ The equation $p''(x)=0$ is equivalent to $(x+a+2)e^x=0$; $x+a+2=0$; $x=-a-2$.

▪ The equation $p'(x)=0$ is equivalent to $(x+a+1)e^x=0$; $x+a+1=0$; $x=-a-1$.

▪ The equation $p(x)=1$ is equivalent to $(x+a)e^x=0$; $x+a=0$; $x=-a$.

These solutions are , in the order $-a-2$; $-a-1$; $-a$, 3 consecutive terms of an increasing arithmetic sequence of common difference 1 .

b) The fourth term of this sequence is $-a+1$; this real number is the solution of the equation $f(x)=e+1$ if and only if $f(-a+1)=e+1$; that is $e^{-a+1}+1=e+1$; $e^{-a+1}=e$; $-a+1=1$; $a=0$.

3- a) $\lim\limits_{x\to-\infty} p(x)=1$ and $\lim\limits_{x\to+\infty} p(x)=+\infty$

$p'(x)=(x+a+1)e^x$.

Table of variations of p

x	$-\infty$		$-a-1$		$+\infty$
$p'(x)$		$-$	0	$+$	
$p(x)$	1	↘	$1-e^{-a-1}$	↗	$+\infty$

Figure 17

The table of variations of p shows that for all a in IR , p has a minimum at $-a-1$.

b) The coordinates of the point S of (γ) corresponding to the minimum of p are $x=-a-1$ and $y=1-e^{-a-1}$ such that , , as a varies , they satisfy the relation $y=1-e^x$.

Therefore, as a varies , the set of S is the curve of equation $y=1-e^x$.

c) The table of variations of p shows that p has an absolute minimum equals to $1-e^{-a-1}$.

Therefore $p(x)\geq 0$ for all x in IR , if and only if $1-e^{-a-1}\geq 0$.

$1-e^{-a-1}\geq 0$ is equivalent to $e^{-a-1}\leq 1$; $-a-1\leq 0$; $a\geq -1$.

d) The functions f and g defined on IR by $f(x)=xe^x+1$ and $g(x)=(x-1)e^x+1$ correspond respectively to $a=0$ and $a=-1$ then , for all x in IR , $f(x)\geq 0$ and $g(x)\geq 0$.

C- The function h is such that $h(x)=\dfrac{xe^x}{xe^x+1}=\dfrac{xe^x}{f(x)}$.

1- a) We proved in part ***B***-1) that , for all x in IR , $f(x)>0$; therefore $f(x)\neq 0$ and h is defined on IR .

b) ▪ For all x in IR , $h(x)=\dfrac{xe^x+1-1}{f(x)}=\dfrac{f(x)-1}{f(x)}=1-\dfrac{1}{f(x)}$.

▪ $\lim\limits_{x\to-\infty} f(x)=1$; therefore $\lim\limits_{x\to-\infty} h(x)=1-1=0$.

▪ $\lim\limits_{x\to+\infty} f(x)=+\infty$; therefore $\lim\limits_{x\to+\infty} h(x)=1-0=1$.

▪ $h(x)=1-\dfrac{1}{f(x)}$; then $h'(x)=\dfrac{f'(x)}{(f(x))^2}$.

$h'(x)$ and $f'(x)$ have the same sign on IR .

Table of variations of h

x	$-\infty$		-1		$+\infty$
$h'(x)$		$-$	0	$+$	
$h(x)$	0	↘	$\frac{1}{1-e}$	↗	1

Figure 21

2- a) $h(0)=0$; therefore (L) passes through the origin of the system .

An equation of the tangent (d) to (L) at this point is $y=h'(0)x$; (d) : $y=x$.

b) For all x in IR , $h(x)-x=\dfrac{xe^x}{f(x)}-x=\dfrac{xe^x-x\,f(x)}{f(x)}=-\dfrac{\left(f(x)-e^x\right)x}{f(x)}=-\left[\dfrac{g(x)}{f(x)}\right]x$.

c) The relative position of (L) and (d) depends on the sign of $h(x)-x$.

The sign of $h(x)-x$ is the opposite to that of x since , for all x in IR , $g(x)>0$ and $f(x)>0$.

- If $x\in\,]-\infty\,;0[$, $h(x)-x>0$ and (L) lies above (d) .
- If $x\in\,]0\,;+\infty[$, $h(x)-x<0$ and (L) lies below (d) .

Since the relative position of (L) and (d) changes at the origin ; therefore this point is a point of inflection of (L) .

$\lim\limits_{x\to-\infty} h(x)=0$ and $\lim\limits_{x\to+\infty} h(x)=1$; then the straight lines of equations $y=0$ and $y=1$ are asymptotes to (L) .

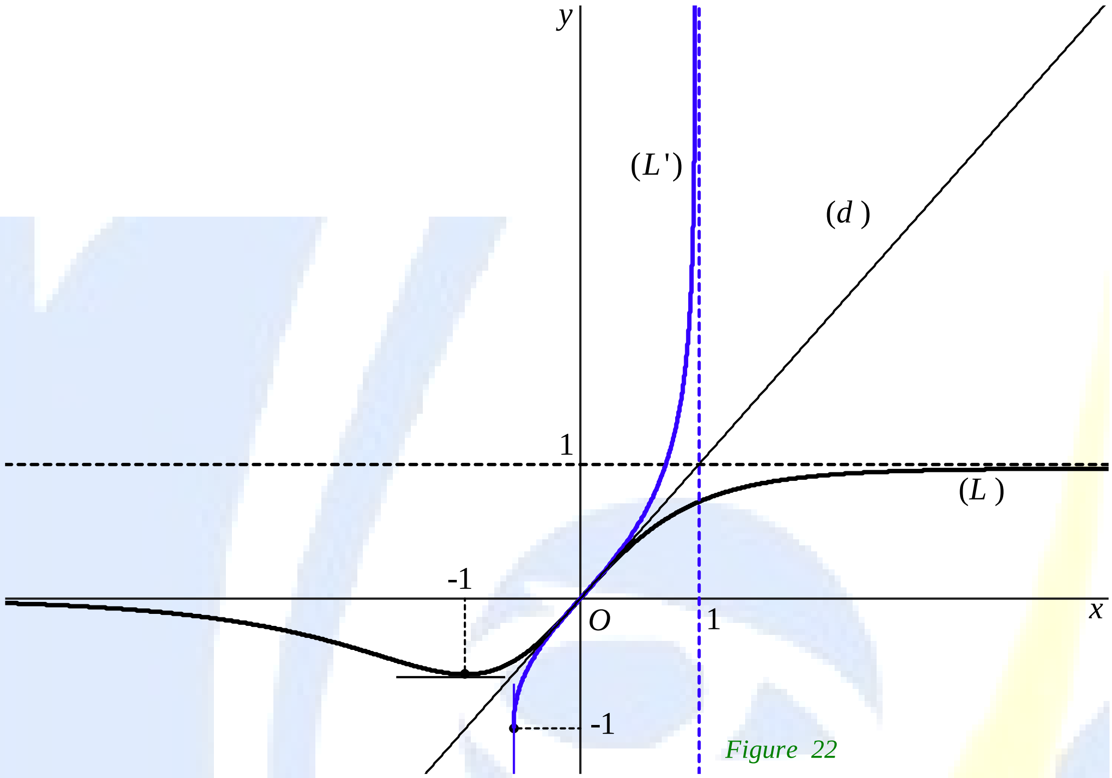


Figure 22

3- a) The restriction of h to the interval $[-1\,;+\infty[$ is continuous and strictly increasing ; therefore , it admits an inverse function h^{-1} defined on $h([-1\,;+\infty[)=[\dfrac{1}{1-e}\,;+\infty[$.

b) The representative curve (L') of h^{-1} is the symmetric of (L) with respect to the straight line (d) of equation $y=x$.

(L) passes through the origin O and is tangent to (d) at this point ; therefore (L') passes through the symmetric of O with respect to (d) which is O it self and is tangent to (d) at this point .

Therefore (L) and (L') are tangent at O.

Drawing (L') by symmetry with respect to (d) .

Exercise 6

A- (δ) and are the straight lines of respective equations $x=-4$ and $x-2y+2=0$.
A point $M(x\,;\,y)$ lies between (δ) and the axis $y'y$ if and only if $x\in[-4\,;\,0]$.

$MK=d(M\,;(\Delta))=\dfrac{|x-2y+2|}{\sqrt{5}}$; $MH=d(M\,;\,y'y)=|x|$; $MH'=d(M\,;(\delta))=|x+4|$.

$M\in(C)$ if and only if M between (δ) and $y'y$ and $5MK^2=3MH\times MH'$; that is
$x\in[-4\,;\,0]$ and $(x-2y+2)^2=3|x(x+4)|$; $(x-2y+2)^2=-3x(x+4)$.
Finally , (C) is the curve of equation $(x-2y+2)^2=-3x(x+4)$

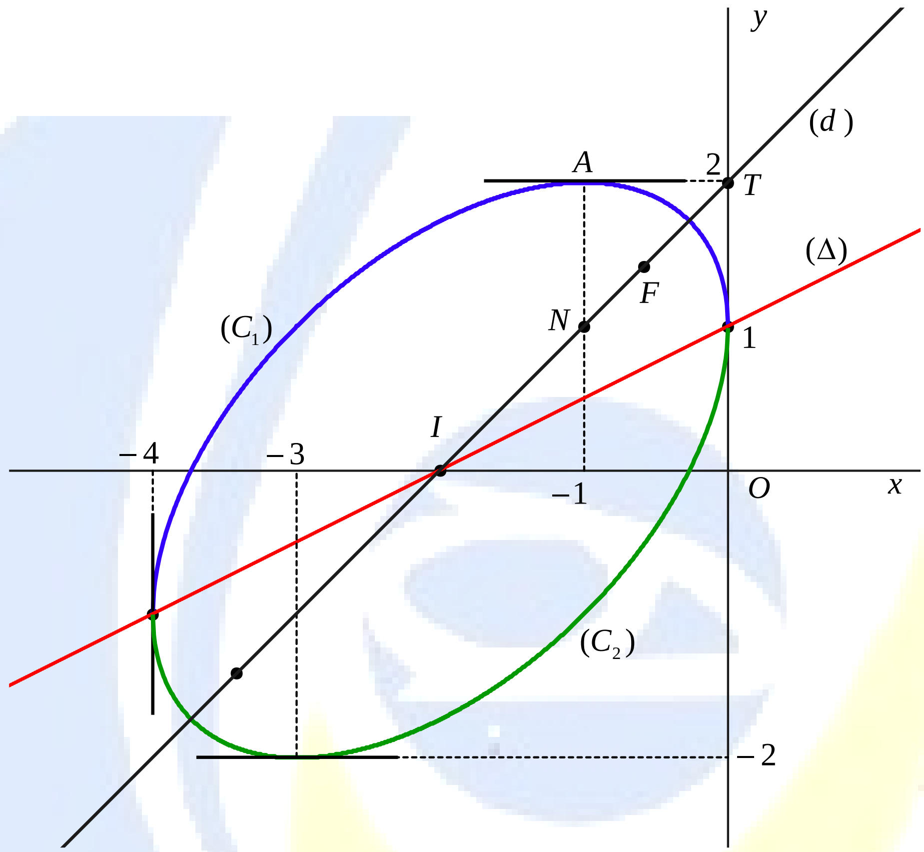
B- (C_1) is the curve of equation of equation $y=\dfrac{1}{2}\left(x+2+\sqrt{-3x^2-12x}\right)$.

1- An equation of (C_2), the symmetric of (C_1) with respect to the point $I(-2\,;\,0)$ is

$$y=-\frac{1}{2}\left(-4-x+2+\sqrt{-3(-4-x)^2-12(-4-x)}\right)=\frac{1}{2}\left(x+2-\sqrt{-3x^2-12x}\right).$$

2- An equation of $(C_1)\cup(C_2)$ is $y=\dfrac{1}{2}\left(x+2+\sqrt{-3x^2-12x}\right)$ *or* $y=\dfrac{1}{2}\left(x+2+\sqrt{-3x^2-12x}\right)$;
that is $2y=x+2\pm\sqrt{-3x^2-12x}$; $2y-x-2=\pm\sqrt{-3x^2-12x}$; $(2y-x-2)^2=-3x^2-12x$;
$(x-2y+2)^2=-3x(x+4)$ which is an equation of (C).
Therefore $(C)=(C_1)\cup(C_2)$.

3- Drawing (C)



C- 1- The complex relation of r is $z' = e^{-i\frac{\pi}{4}} z$ which is equivalent to $z = e^{i\frac{\pi}{4}} z'$; that is

$x + i\,y = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)(x' + i\,y')$; therefore $x = \frac{\sqrt{2}}{2}(x' - y')$ and $y = \frac{\sqrt{2}}{2}(x' + y')$.

The equation of (C) can be written as $x^2 - xy + y^2 + 4x - 2y + 1 = 0$.

By substituting for x and y, the equation $x^2 - xy + y^2 + 4x - 2y + 1 = 0$ becomes

$$\frac{1}{2}(x' - y')^2 - \frac{1}{2}(x'^2 - y'^2) + \frac{1}{2}(x' + y')^2 + 2\sqrt{2}(x' - y') - \sqrt{2}(x' + y') + 1 = 0 \ ;$$

$$\frac{1}{2}x'^2 + \frac{3}{2}y'^2 + \sqrt{2}\,x' - 3\sqrt{2}\,y' + 1 = 0 \ ; \quad x'^2 + 3y'^2 + 2\sqrt{2}\,x' - 6\sqrt{2}\,y' + 2 = 0 \ .$$

Therefore , an equation of (E) is $x^2 + 3y^2 + 2\sqrt{2}\,x - 6\sqrt{2}\,y + 2 = 0$.

2- a) The equation $x^2+3y^2+2\sqrt{2}\,x-6\sqrt{2}\,y+2=0$ can be written as

$(x+\sqrt{2})^2-2+3\,(y-\sqrt{2})^2-6+2=0$; $(x+\sqrt{2})^2+3\,(y-\sqrt{2})^2=6$; $\dfrac{(x+\sqrt{2})^2}{6}+\dfrac{(y-\sqrt{2})^2}{2}=1$.

Therefore (E) is an ellipse .

For the ellipse (E) , $a=\sqrt{6}$ and $b=\sqrt{2}$; the area of (E) is $S=\pi ab$ units of area .

$S=\pi\sqrt{6}\times\sqrt{2}=2\sqrt{3}\pi$ *units of area* ; that is $S=8\sqrt{3}\pi \quad cm^2$.

b) The rotation preserves the nature of a conic and (E) is an ellipse then (C) is also an ellipse .

The point I is the center of (C) since (C) is formed of two parts (C_1) and (C_2) symmetric with respect to I .

The rotation preserves areas then the area of (C) is also equal to $8\sqrt{3}\pi \quad cm^2$.

The area of (C) is the area of the domain bounded by (C_1) and (C_2) ; since (C_1) is above (C_2)

then $S=\displaystyle\int_{-4}^{0}\frac{1}{2}\left(x+2+\sqrt{-3x^2-12x}-x-2+\sqrt{-3x^2-12x}\right)dx=\int_{-4}^{0}\sqrt{-3x^2-12x}\,dx$ *units of area* .

Consequently , $\displaystyle\int_{-4}^{0}\sqrt{-3x^2-12x}\,dx=2\sqrt{3}\pi$.

c) The focal axis of (E) is the straight line (δ) of equation $y=\sqrt{2}$; that of (C) is the straight line (d) whose image by the rotation r is (δ).

The complex relation of r is $z'=e^{-i\frac{\pi}{4}}z$; $x'+iy'=\left(\dfrac{\sqrt{2}}{2}-i\dfrac{\sqrt{2}}{2}\right)(x+iy)$ then $y'=\dfrac{\sqrt{2}}{2}(y-x)$

Therefore an equation of (d) is $\dfrac{\sqrt{2}}{2}(y-x)=\sqrt{2}$ which is $y=x+2$.

For the ellipse (E) , $c=\sqrt{a^2-b^2}=\sqrt{6-2}=2$ then $F'(x'=2-\sqrt{2}\ ;\ y'=\sqrt{2})$.

The point F such that $r(F)=F'$ is a focus of (C) .

The coordinates of F are $x=\dfrac{\sqrt{2}}{2}(x'-y')=\sqrt{2}-2$ and $y=\dfrac{\sqrt{2}}{2}(x'+y')=\sqrt{2}$.

OR The rotation preserves the distances then $IF=c=2$.

Therefore the foci of (C) are the points on (d) such that $IF=2$; that is

$F(x\ ;\ x+2)$ and $IF^2=(x+2)^2+(x+2)^2=4$; therefore $(x+2)^2=2$; $x+2=\sqrt{2}$ or $x+2=-\sqrt{2}$.

The foci of (C) are the points $(-2+\sqrt{2}\ ;\ \sqrt{2})$ and $(-2-\sqrt{2}\ ;\ -\sqrt{2})$