LEBANESE UNIVERSITY
FACULTY OF ENGINEERING

الجامعة اللبنانية
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Entrance Exam 2009 - 2010 **Mathematics** **Duration : 3 hours**
July 11 , 2009

The distribution of grades is over 25

I- (5 pts) ***A-*** Given the table of variations of a continuous function g defined on $]0\,;+\infty[$ by $g(x)=m+n\dfrac{\ell n\,x}{x}$.

x	0		e		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$	$+\infty$	$\searrow$	$1-2e^{-1}$	$\nearrow$	1

1- a) Prove that $m=1$ and $n=-2$.

b) Prove that , for all x in $]0\,;+\infty[$, $\ell nx\le\dfrac{x}{e}$.

2- a) Prove that the representative curve of any antiderivative of g on $]0\,;+\infty[$ admits a point of inflection I .

b) Determine the antiderivative G of g for which the point I belongs to the line of equation $y=x$.

B- Let F be the function such that $F(x)=\dfrac{x}{x-\ell n\,x}$.

1- a) Using part ***A*** , justify that F is defined on $]0\,;+\infty[$.

b) Prove that F admits an extension by continuity at 0 and define the extension function f of F .

c) Prove that f is differentiable at 0 .

2- Let (U_n) be the sequence defined for $n\in IN$, by $U_n=\left(\dfrac{\ell n\,a}{a}\right)^n$ where a is a *given real* number in $]1\,;+\infty[$.

a) Prove that (U_n) is a strictly decreasing geometric sequence .

b) Let S_n be the sum defined by $S_n=U_0+U_1+U_2+U_3+\cdots\cdots\cdots\cdots+U_n$.

Calculate S_n in terms of n and a , then prove that $\lim\limits_{n\to+\infty}S_n=f(a)$.

II- (3 pts) The staff of a hospital is distributed into three categories : Doctors (D) , Nurses (N) and Technicians (T) .

$20\ \%$ are doctors and $50\ \%$ are nurses .

$75\ \%$ of the doctors are men and $80\ \%$ of the nurses are women .

We ask randomly one member of the staff .

1- Calculate the probability that this person is :

a) a technician ; b) a woman knowing that she is a doctor ; c) a man knowing that he is a nurse .

2- Calculate the probability that this person is :

a) a woman doctor ; b) a woman nurse .

3- Knowing that $51\,\%$ of the staff are women .

a) Calculate the probability that the asked person is a woman technician .

b) Deduce the probability that the asked person is a woman knowing that she is a technician .

III - (6 pts) ***A***- 1- Solve the differential equation $(I):\ y' + 2xy = 0$ and prove that its general solution can be written in the form $y = Ce^{-x^2}$ where C is an arbitrary constant.

2- Consider the differential equation $(II):\ xy' + 2(x^2-1)y = 0$.

Let $y = x^2 z$ where z is a differentiable function defined on IR^*.

a) Determine a differential equation whose general solution is the function z.

b) Determine the function z and deduce the general solution of the equation (II).

B- Consider the functions f and g defined on IR by $f(x) = e^{-x^2}$ and $g(x) = x^2 e^{-x^2}$.

Designate by (C) the representative curve of f and by (C') that of g.

1- Prove that f is an even function and set up its table of variations.

2- Prove that g is an even function and set up its table of variations.

3- a) Determine the points of intersection of (C) and (C').

b) Draw (C) and (C') in the same orthonormal system $(O;\ \vec{i},\ \vec{j})$ (*Graph unit* : 3 *cm*)

4- Let F be the antiderivative of f on IR that satisfies $F(0) = 0$ and G the function defined on IR by

$G(x) = \frac{1}{2}\left[F(x) - xe^{-x^2}\right]$. Prove that G is the antiderivative of g on IR that satisfies $G(0) = 0$.

5- Given that $F(1) = 0.75$.

a) Calculate the area A of the domain bounded by (C), the axis of abscissas and the two straight lines of equations $x = -1$ and $x = 1$.

b) Calculate the area A' of the domain bounded by (C), (C') and the straight lines of equations $x = -1$ and $x = 1$.

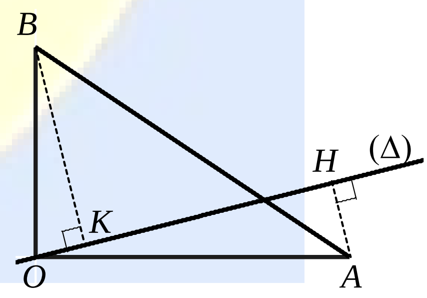
6- Let S be the area of the domain bounded by (C) and the semi straight lines $[Ox)$ and $[Oy)$, and S' the area of the domain bounded by (C') and the semi straight lines $[Ox)$ and $[Oy)$. Prove that $S = 2S'$.

IV- (5 pts) In the oriented plane, consider a right triangle AOB such that $(\overrightarrow{OA}, \overrightarrow{OB}) = \frac{\pi}{2}\quad (2\pi)$.

Let (Δ) be a variable straight line passing through O.

H and K are the orthogonal projections of A and B on (Δ).

Let S be the similitude such that $S(O) = A$ and $S(B) = O$.



1- Determine the angle of S.

2- Prove that the center I of S belongs to the circles of diameters $[OA]$ and $[OB]$. Deduce that I is the orthogonal projection of O on $[AB]$.

3- a) Determine the image by S of each of (BK) and (Δ).

Deduce that $S(K) = H$.

b) Prove that, as (Δ) varies, the circle (γ) of diameter $[HK]$ passes through a fixed point to be determined.

4- Consider the dilation (homothecy) h of center B and ratio 2 .
Let M be the mid point of $[OB]$; O' and B' the symmetric of O and B respectively with respect to I .
a) Prove that $S(B') = O'$ and determine $S \circ h(M)$ and $S \circ h(I)$.
b) Deduce that the median (IM) in triangle IOB is a height in triangle IAO'.

V- (6 pts) In the plane referred to a direct orthonormal system $(O\,;\,\vec{i}\,,\,\vec{j}\,)$, consider the parabola (P) of equation $y^2 = 4(x+1)$.

1- a) Determine the focus , the directrix (d) and the vertex V of (P).
b) Draw (P) and the tangent (Δ) to (P) at V .

2- Let A be a point of (P) of ordinate a $(a \neq 0)$, A' the orthogonal projection of A on (Δ) and (D) the perpendicular to (VA) passing through A' .
a) Write an equation of (D) and prove that , as A varies on (P), (D) passes through a fixed point L to be determined .
b) (D) cuts (VA) at E . Prove that , as A varies on (P), E varies on a fixed circle to be determined .

3- The straight line (OA) cuts the parabola (P) again at B . Let I be the mid point of $[AB]$.
Designate by C , D and J the respective orthogonal projections of A , B and I on (d) .
a) Calculate IJ in terms of AC and BD .
b) Prove that , when A varies on (P), the circle (γ) of diameter $[AB]$ remains tangent to (d) .

4- a) Let b be the ordinate of B . Prove that $ab = -4$.
b) The normal at A to (P) and the normal at B to (P) intersect at N . Prove that N belongs to (γ) .

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I- *A-* 1- The function g is defined on $]0\,;+\infty[$ by $g(x)=m+n\dfrac{\ell n\,x}{x}$.

a) $\lim\limits_{x\to+\infty} g(x)=m=1$ and $g(e)=m+\dfrac{n}{e}=1+\dfrac{n}{e}=1-\dfrac{2}{e}$. Therefore $n=-2$.

Finally , $g(x)=1-2\dfrac{\ell n\,x}{x}$.

b) The given table shows that , for all x in $]0\,;+\infty[$, $1-2\dfrac{\ell n\,x}{x}\geq 1-\dfrac{2}{e}$; $\ell nx\leq\dfrac{x}{e}$ ($x>0$) .

2- a) $G'(x)=g(x)$ and $G''(x)=g'(x)$.

The sign of $g'(x)$ changes at e ; therefore , the concavity of the curve of G changes at the point I of abscissa e . Therefore , (C) has a point of inflection $I(e\,;G(e))$.

b) $G(x)=\displaystyle\int g(x)\,dx=\int[1-2\frac{\ell n\,x}{x}]\,dx=x-\ell n^2x+C$.

I belongs to the line of equation $y=x$ if and only if $G(e)=e$; then $C=1$. Finally $G(x)=x+1-\ell n^2 x$.

B- 1- a) For all x in $]0\,;+\infty[$, $\ell nx\leq\dfrac{x}{e}<x$. Therefore $x-\ell n\,x\neq 0$ and F is defined on $]0\,;+\infty[$.

b) ▪ The function $x\to x-\ell n\,x$ is continuous on $]0\,;+\infty[$; then F is continuous on $]0\,;+\infty[$.

▪ $\lim\limits_{x\to0^+}[x-\ell n\,x]=+\infty$; therefore $\lim\limits_{x\to0^+}F(x)=\lim\limits_{x\to0^+}\dfrac{x}{x-\ell n\,x}=0$ (finite limit).

Therefore F admits an extension by continuity at 0 .
The extension function f is defined on $[0\,;+\infty[$ by

$$\begin{cases} f(0)=0 \\ f(x)=\dfrac{x}{x-\ell n\,x} \quad \textit{for } x\neq 0\end{cases}$$

c) $\lim\limits_{x\to0^+}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0^+}\dfrac{1}{x-\ell n\,x}=0$ (finite limit) . Therefore f is differentiable at 0 and $f'(0)=0$.

2- a) ▪ $U_{n+1}=\left(\frac{\ell n\, a}{a}\right)^{n+1}=\left(\frac{\ell n\, a}{a}\right)^{n}\times\left(\frac{\ell n\, a}{a}\right)=U_n\times\left(\frac{\ell n\, a}{a}\right)$; therefore (U_n) is a geometric sequence of common ratio $r=\frac{\ell n\, a}{a}$ and first term $U_0=1$.

▪ $U_{n+1}-U_n=\left(\frac{\ell n\, a}{a}\right)^{n+1}-\left(\frac{\ell n\, a}{a}\right)^{n}=\left(\frac{\ell n\, a}{a}\right)^{n}\times\left(\frac{\ell n\, a}{a}-1\right)$ where $\frac{\ell n\, a}{a}>0$ and $\frac{\ell n\, a}{a}\le\frac{1}{e}<1$.

Therefore , $U_{n+1}-U_n<0$ and (U_n) is strictly decreasing .

b) Since (U_n) is a geometric sequence and S_n is the sum of $n+1$ consecutive terms , then

$$S_n=U_0+U_1+U_2+U_3+\cdots\cdots\cdots\cdots+U_n=U_0\times\frac{1-r^{n+1}}{1-r}=\frac{1-\left(\frac{\ell n\, a}{a}\right)^{n+1}}{1-\frac{\ell n\, a}{a}}.$$

Since $0<\frac{\ell n\, a}{a}<1$, $\lim\limits_{n\to+\infty}\left(\frac{\ell n\, a}{a}\right)^{n+1}=0$; therefore

$$\lim_{n\to+\infty}S_n=\frac{1}{1-\frac{\ell n\, a}{a}}=\frac{a}{a-\ell n\, a}=f(a)$$

II- 1- When a member of the staff is selected at random, there are three possibilities: a doctor (D) , a nurse (N) or a Technician (T) .

a) It is given that $p(D)=0.2$, $p(N)=0.5$ then $p(T)=1-0.2-0.5=0.3$.

b) For each one of the three categories, there are two possibilities: man (M) or woman (W) .

It is given that $p(M/D)=0.75$ then, $p(W/D)=p(\overline{M}/D)=1-0.75=0.25$.

c) It is given that $p(W/N)=0.8$ then , $p(M/N)=p(\overline{W}/N)=1-0.8=0.2$.

2- a) The event " the person is a woman doctor " can be represented by $D\cap W$; its probability is $p(D\cap W)=p(D)\times p(W/D)=0.2\times0.25=0.05$.

b) The event " the person is a woman nurse " can be represented by $N\cap W$; its probability is $p(N\cap W)=p(N)\times p(W/N)=0.5\times0.8=0.4$.

3- a) The event " the person is a woman technician " can be represented by $T\cap W$.

By the formula of total probability , $p(W)=p(D\cap W)+p(N\cap W)+p(T\cap W)$, therefore $p(T\cap W)=p(W)-p(D\cap W)-p(N\cap W)$

If $51\,\%$ of the staff are women then $p(W)=0.51$. Thus
$p(T\cap W)=0.51-0.05-0.4=0.06$.

b) The probability that the asked person is a woman knowing that she is a technician is equal to
$p(W/T)=\dfrac{p(W\cap T)}{p(T)}=\dfrac{0.06}{0.3}=0.2$.

III - *A*- 1- (I) : $y'+2xy=0$.

▪ The function $y=0$ is a particular solution of (I).

▪ The other solutions are those of the equation $\dfrac{y'}{y}=-2x$ then, $\ell n|y|=-x^2+K$; $K\in IR$.

$|y|=e^K\times e^{-x^2}$; $|y|=a\,e^{-x^2}$ where $a\in]0+\infty[$; $y=\lambda e^{-x^2}$ where $\lambda\in IR^*$.

The general solution of (I) is $\begin{cases} y=0 \quad and \\ y=\lambda e^{-x^2} \text{ where } \lambda\in IR^* \end{cases}$ which is $y=C\,e^{-x^2}$ where $C\in IR$.

2- Consider the differential equations (II) : $x\,y'+2(x^2-1)\,y=0$.

a) If $y=x^2\,z$ then , $y'=2x\,z+x^2\,z'$.

By substitution in the equation (II) we obtain $2x^2\,z+x^3\,z'+2(x^2-1)x^2\,z=0$; that is $z'+2x\,z=0$.

b) According to part 1) , the general solution of the equation $z'+2x\,z=0$ is $z=C\,e^{-x^2}$.

Therefore , the general solution of the equation (II) is $y=C\,x^2\,e^{-x^2}$ where $C\in IR$.

B- 1- ▪ The set IR is centered at 0 and , for all x in IR , $f(-x)=f(x)$; therefore the function f is even .

▪ $\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to+\infty} f(x)=0$;

$f'(x)=-2x\,e^{-x^2}$.

x	$-\infty$		0		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	0	$\nearrow$	1	$\searrow$	0

2- ▪ The set IR is centered at 0 and, for all x in IR, $g(-x)=g(x)$; therefore the function g is even .

x	$-\infty$		-1		0		1		$+\infty$
$g'(x)$		$+$	0	$-$	0	$+$	0	$-$	
$g(x)$	0	$\nearrow$	e^{-1}	$\searrow$	0	$\nearrow$	e^{-1}	$\searrow$	0

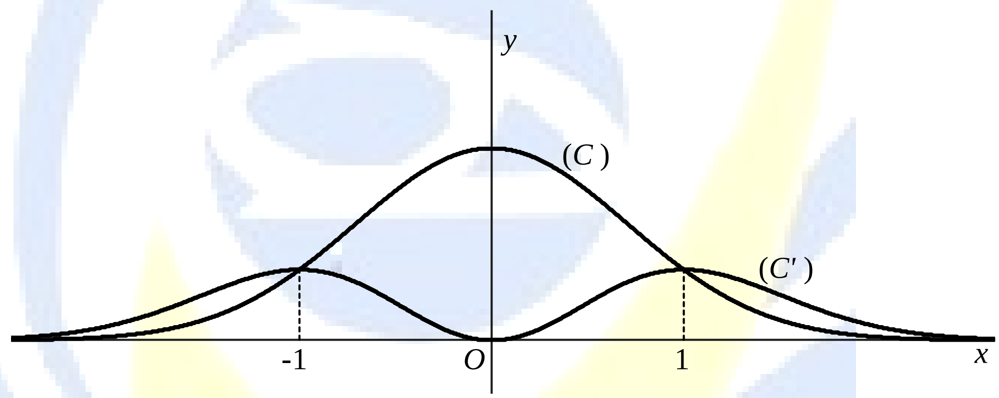
▪ $\lim\limits_{x\to-\infty} g(x)=\lim\limits_{x\to+\infty} g(x)=\lim\limits_{t\to+\infty}\dfrac{t}{e^t}=0$.

$g'(x)=2xe^{-x^2}-2x^3e^{-x^2}=2x(1-x^2)e^{-x^2}$.

3- a) The abscissas of the points of intersection of (C) and (C') are the roots of the equation $f(x)=g(x)$; $x^2-1=0$; $x=-1$ or $x=1$.

The points of intersection of (C) and (C') are $(-1\,;e^{-1})$ and $(1\,;e^{-1})$.

b) Drawing (C) and (C').



4- The function F is defined on IR that satisfies $F'(x)=f(x)=e^{-x^2}$ and $F(0)=0$.

G is the function defined on IR by $G(x)=\dfrac{1}{2}\left[F(x)-xe^{-x^2}\right]$.

▪ $G'(x)=\dfrac{1}{2}\left[F'(x)-e^{-x^2}+2x^2e^{-x^2}\right]=\dfrac{1}{2}\left[e^{-x^2}-e^{-x^2}+2x^2e^{-x^2}\right]=x^2e^{-x^2}=g(x)$.

▪ $G(0)=\dfrac{1}{2}\left[F(0)-0\right]=0$.

Therefore , G is the antiderivative of g on IR that satisfies $G(0)=0$.

5- a) The curve (C) lies above the axis of abscissas then , the required area is $A=\displaystyle\int_{-1}^{1} f(x)\,dx$ *units of*

The function is even then,

$$\int_{-1}^{1} f(x)\,dx = 2\int_{0}^{1} f(x)\,dx = 2\big[\,F(x)\,\big]_0^1 = 2[\,F(1)-F(0)\,] = 2[\,0.75-0\,] = 1.5\ .$$

Finally, $A = 1.5$ *units of area*; that is $A = 1.5\times 3^2 = 13.5\quad cm^2$.

b) In the interval $[-1\,;1]$, (C) lies above (C') then , the required area is $A' = \int_{-1}^{1}[\,f(x)-g(x)\,]dx$ *units of area.*

$$\int_{-1}^{1}[\,f(x)-g(x)\,]dx = 2\int_{0}^{1}[\,f(x)-g(x)\,]dx = 2\big[\,F(x)-G(x)\,\big]_0^1 = \Big[F(x)+x\,e^{-x^2}\Big]_0^1 = F(1)+e^{-1} = 0.75+e^{-1}$$

Finally , $A' = 0.75+e^{-1}$ *units of area* ; that is $A = 6.75+9e^{-1}\quad cm^2$.

6- Let m be a strictly positive number .

In the interval $[\,0\,;\,m\,]$, (C) lies above the axis of abscissas then , the area of the domain bounded by (C), $x'x$, $y'y$ and the straight line of equation $x = m$ is equal to $I(m)$ *units of area* .

$$I(m) = \int_{0}^{m} f(x)\,dx = F(m)-F(0) = F(m)\,.$$

Therefore $S = \lim\limits_{m\to+\infty} I(m) = \lim\limits_{m\to+\infty} F(m) = \lim\limits_{x\to+\infty} F(x)$ *units of area* .

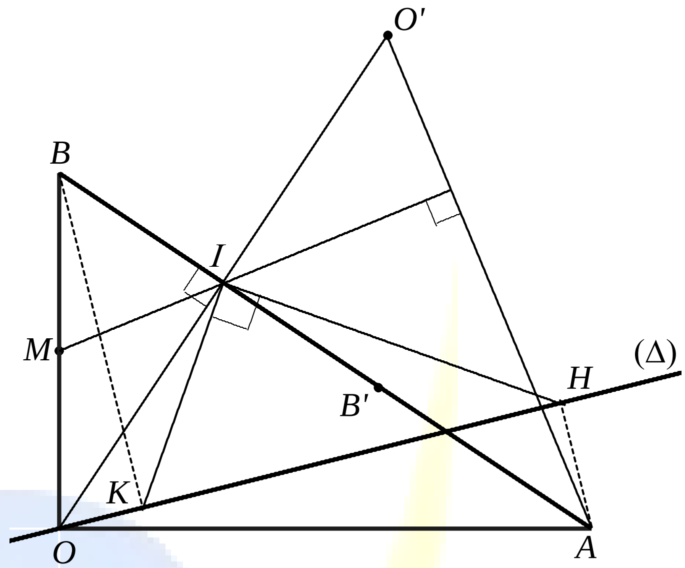
Similarly , $S' = \lim\limits_{x\to+\infty} G(x)$ *units of area* .

We have $G(x) = \dfrac{1}{2}\Big[F(x)-x\,e^{-x^2}\Big]$; then $\lim\limits_{x\to+\infty} G(x) = \dfrac{1}{2}\lim\limits_{x\to+\infty} F(x)$ since $\lim\limits_{x\to+\infty} x\,e^{-x^2} = 0$.

Therefore $S' = \dfrac{1}{2}S$; that is $S = 2S'$.

IV- 1- The similitude S is such that $S(O) = A$ and $S(B) = O$.

The angle of S is $(\overrightarrow{OB}\ ;\ \overrightarrow{AO}\,) = \pi + (\overrightarrow{OB}\ ;\ \overrightarrow{OA}\,) = \pi - \dfrac{\pi}{2} = \dfrac{\pi}{2}\quad (2\pi)$.



2- I is the center of S .

- $S(O) = A$; then $(\overrightarrow{IO}\ ;\ \overrightarrow{IA}) = \frac{\pi}{2}$ and I belongs to the circle of diameter $[OA]$.
- $S(B) = O$; then $(\overrightarrow{IB}\ ;\ \overrightarrow{IO}) = \frac{\pi}{2}$ and I belongs to the circle of diameter $[OB]$.

The only common point , other than O, of these two circles is the orthogonal projection of O on $[AB]$, then the center I of S is the orthogonal projection of O on $[AB]$.

3- a) The angle of S is $\frac{\pi}{2}$; then any straight line and its image by S are perpendicular .

- $S(B) = O$; then the image by S of (BK) is the perpendicular to (BK) passing through O which is the straight line (Δ).
- $S(O) = A$; then the image by S of (Δ) is the perpendicular to (Δ) passing through O which is the straight line (AH).
- K is the point of intersection of (BK) and (Δ); the image of K by S is the point of intersection of (Δ) and (AH) ; that is $S(K) = H$.

b) $S(K) = H$; then , $(\overrightarrow{IK}\ ;\ \overrightarrow{IH}) = \frac{\pi}{2}$. Therefore , as (Δ) varies , the circle (γ) of diameter $[HK]$ passes through the fixed point I .

4- a) ▪ O' and B' the symmetric of O and B with respect to I ; then $\overrightarrow{IO'} = -\overrightarrow{IO}$ and $\overrightarrow{IB'} = -\overrightarrow{IB}$.

$S(B) = O$; then , $IO = \lambda IB$ (λ is the ratio of S) and $(\overrightarrow{IB}\ ;\ \overrightarrow{IO}) = \frac{\pi}{2}$; therefore

$IO' = \lambda IB'$ and $(\overrightarrow{IB'}\ ;\ \overrightarrow{IO'}) = \frac{\pi}{2}$. consequently , $S(B') = O'$.

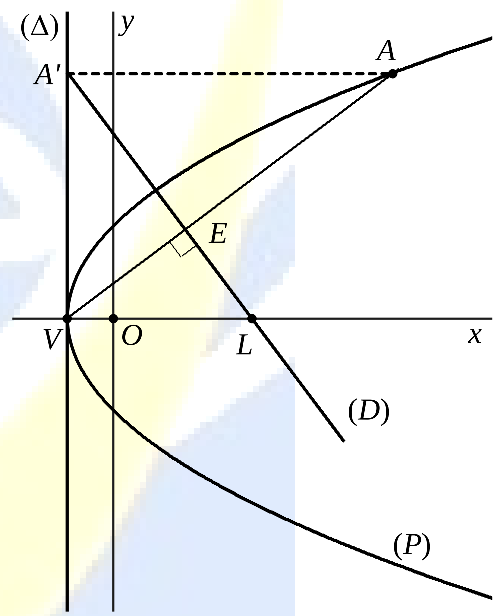
- M is the mid point of $[OB]$; then $\overrightarrow{BO} = 2\ \overrightarrow{BM}$ and $h(M) = O$.
- $S \circ h(M) = S\big(h(M)\big) = S(O) = A$ and $S \circ h(I) = S\big(h(I)\big) = S(B') = O'$.

b) $S\circ h(M)=A$ and $S\circ h(I)=O'$. Therefore the image of the median (IM) in triangle IOB is the straight line (AO').

But h is a positive dilation , then $S\circ h$ is a similitude of same angle $\frac{\pi}{2}$; therefore the straight line (IM) and its image (AO') by S are perpendicular .
finally , the median (IM) in triangle IOB is a height in triangle IAO'.

V- 1- a) $(P):\ y^2=4(x+1)$.
The parameter of (P) is $p=2$, the vertex is $V(-1\,;\,0)$, the focus is $O(0\,;\,0)$ and the directrix is the straight line (d) of equation $x=-2$.

b) $(\Delta):\ x=-1$
Drawing (P) and (Δ).



2- $A(\frac{a^2}{4}-1\,;\,a)$; $A'(-1\,;\,a)$.

a) (D) is the perpendicular to (VA) through A' ;

$\overrightarrow{VA}(\frac{a^2}{4}\,;\,a)$ is a normal vector to (D).

$(D):\ \frac{a^2}{4}(x+1)+a(y-a)=0$; $a(x-3)+4y=0$.

As A varies on (P) , a traces IR^*
and (D) passes through the fixed point $L(3\,;\,0)$.

b) $L\hat{E}V=90^\circ$ where L and V are fixed .
therefore , as A varies on (P) , E
varies on the fixed circle of diameter $[LV]$.

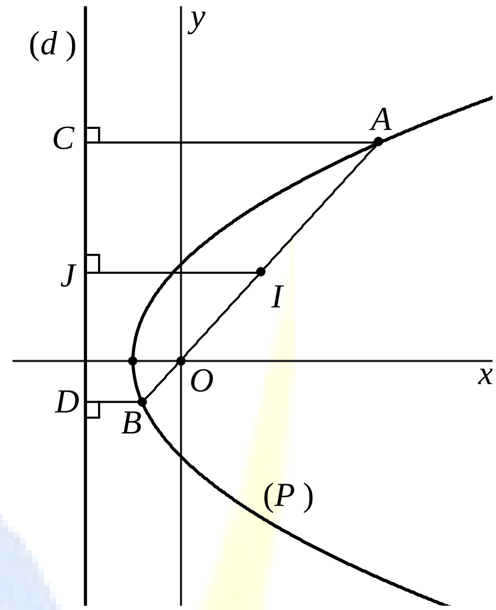
3- The radius of the circle of diameter $[AB]$ is $r=\frac{1}{2}AB$.

a) The distance of I from (d) is equal to $IJ=\frac{1}{2}(AC+BD)$.

b) A and B are on the parabola (P) then $AC = AO$ and $BD = BO$; therefore , the distance from I to (d) is

$IJ = \frac{1}{2}(AO + BO) = \frac{1}{2}AB = r$.

Hence , when A varies on (P), the circle of diameter $[AB]$ remains tangent to (d).



4- $A(\frac{a^2}{4} - 1 \,;\, a)$ and $B(\frac{b^2}{4} - 1 \,;\, b)$.

- A, O and B are collinear ; therefore $\det(\overrightarrow{OA} \,;\, \overrightarrow{OB}) = 0$.

  This gives $\frac{a^2 b}{4} - b - \frac{b^2 a}{4} + a = 0$; $(\frac{ab}{4} + 1)(a - b) = 0$.

  Therefore $\frac{ab}{4} + 1 = 0$ and $ab = -4$

- The equation $y^2 = 4(x+1)$ gives $2y\,y' = 4$.

  The slope of the tangent at A is $y'_A = \frac{2}{a}$

  The slope of the normal at A is $-\frac{a}{2}$ and that of the normal at B is $-\frac{b}{2}$

  $\left(-\frac{a}{2}\right) \times \left(-\frac{b}{2}\right) = \frac{ab}{4} = -1$; therefore these two lines are perpendicular and $A\hat{N}B = 90^\circ$.

  Hence $N \in (\gamma)$.