LEBANESE UNIVERSITY
FACULTY OF ENGINEERING

الجامعة اللبنانية
كلية الهندسة

Entrance Exam 2008-2009 **Duration: 3 hours**

MATHEMATICS

The grades are over 25

I- (2.5 points) Let f be a differentiable function defined on the interval $I =]0\,;+\infty[$ such that $f(I) = IR$ and for all a and b in I, $f(a\times b) = f(a) + f(b)$.

1- Prove that $f(1)=0$. Deduce that, for all x in I, $f(\frac{1}{x}) = -f(x)$.

2- For all a and b in I, express $f(\frac{a}{b})$ in terms of $f(a)$ and $f(b)$.

3- Suppose, in addition, that for all x in $]0\,;1[$, $f(x)<0$.

a) Prove that f is strictly increasing on I. Justify that f has an inverse function.

b) Prove that, for all a and b in IR, $g(a+b) = g(a)\times g(b)$ and express $g(-a)$ in terms of $g(a)$.

II- (3.5 points) We are given 3 urns U, V and W containing each n identical balls ($n \in IN$ and $n \geq 4$) such that :

Four balls of the urn U are red and the others are white; one ball of the urn V is red and the others are white; two balls of the urn W are red and the others are white .

The game consists in rolling a perfect die , then

- If the die shows 4, the player draws a ball at random from U.
- If the die shows an even number different than 4 , the player draws a ball at random from V.
- If the die shows an odd number, the player draws a ball at random from W.

Consider the events : A : " the die shows 4 " ; B : " the die shows an even number other than 4 " .
C : " the die shows an odd number " and R : " the drawn ball is red ".

1- a) Calculate the conditional probabilities $p(R/A)$, $p(R/B)$ and $p(R/C)$ in terms of n .

b) Prove that $p(R) = 2/n$ and that the events C and R are independent.

2- a) Calculate the probability that the die shows the number 4 knowing that the drawn ball is red .

b) Calculate the probability that the die shows an odd number knowing that the drawn ball is white.

3- a) Determine n so that $p(R) > 0.4$.

b) Determine n so that the probability the drawn ball is white is double the probability that the drawn ball is red.

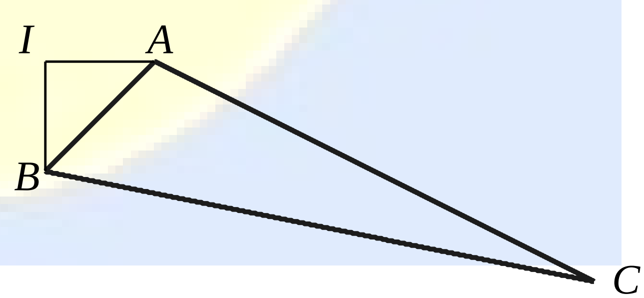
4- Suppose that $n=6$. The game is repeated 10 times by replacing, each time, the drawn ball in the urn.
Calculate the probability of the event " at least one of the 10 drawn balls is red " .

III- (5.5 points) Consider a direct triangle ABC . Let I be the point such that IBA is direct and right isosceles at I

1- Let R be the rotation of center I that transforms A into B .

a) Determine the angle of R .

b) Construct the image D of C by R and determine the principal measure of $(\overrightarrow{AC}\,;\overrightarrow{BD})$.



2- a) Construct the center J of the rotation R' of angle $\frac{\pi}{2}$ that transforms A into D .

b) Prove that $R'(C) = B$.

3- Let M be the midpoint of $[AC]$ and N that of $[BD]$. Prove that $IMJN$ is a square.

4- Let P and Q be the points such that $IAPB$ and $ICQD$ are two indirect squares.

a) Determine the ratio and the angle of the similitude S of center I that transforms A into P.

b) Determine $S(C)$ and $S(M)$. What can be deduced for the point J ?

5- Suppose in this part that the plane is referred to a direct orthonormal system $(O\,;\,\vec{u}\,,\,\vec{v})$ and that the affixes of A, B, C are $z_A = 2-i$, $z_B = 1-2i$, $z_C = 6-3i$.

a) Determine the complex relation of the rotation R and deduce the affix of its center I.

b) Determine the affix of D.

c) Determine the complex relation of the similitude S.

d) Determine the affix of each of the points J, P and Q and verify that J is the midpoint of $[PQ]$.

IV- (5.5 points) The plane is referred to a direct orthonormal system $(O\,;\,\vec{i}\,,\,\vec{j})$.

A- Consider the point $M(4\cos\alpha\,;\,2\cos 2\alpha)$, where α is a real number.

Prove that, as α varies, the set of M is a part of a parabola to be determined.

B- Let (P) be the parabola of equation $x^2 = 4(y+2)$.

1- Determine the focus F and the directrix (d) of (P). Draw (P).

2- Consider the points A and B of (P) with abscissas α and β respectively.

Let (d_1) be the tangent to (P) at A and (d_2) be the tangent to (P) at B.

a) Determine an equation of each of the straight lines (d_1) and (d_2).

b) Prove that (d_1) and (d_2) intersect at the point $T\left(\dfrac{\alpha+\beta}{2}\,;\,\dfrac{\alpha\beta-8}{4}\right)$.

3- Prove that if (d_1) and (d_2) are perpendicular then T belongs to the directrix (d) of (P) and (AB) passes through the focus F of (P).

4- Suppose that (d_1) and (d_2) are not perpendicular and let θ be a measure of their acute angle.

Prove only one of the two relations: $\cos\theta = \dfrac{|\alpha\beta+4|}{\sqrt{(4+\alpha^2)(4+\beta^2)}}$ or $\tan\theta = 2\left|\dfrac{\alpha-\beta}{4+\alpha\beta}\right|$.

5- a) Prove that, if α and β vary such that $\theta = 45^\circ$, then $4(\alpha+\beta)^2 - (\alpha\beta+12)^2 = -128$.

Deduce that T varies on a hyperbola (H) whose equation is to be determined.

b) Prove that the focus F and the directrix (d) of (P) are also a focus and a directrix of (H).

V- (8 points ***A***- Consider the function f defined on $[0\,;\,+\infty[$ by $f(0)=0$ and $f(x) = x(\ell n\, x)^2 - 2x\,\ell n\, x + 2x$ for $x \neq 0$.

Let (C) be the representative curve of f in an orthonormal system $(O\,;\,\vec{i}\,,\,\vec{j})$.

1- a) Prove that f is continuous at 0.

b) Calculate $\lim\limits_{x\to 0^+} \dfrac{f(x)}{x}$. Deduce the tangent to (C) at the point $O(0\,;\,0)$.

2- Set up the table of variations of f.

3- Determine an equation of the tangent (d) to (C) at the point E of abscissa e. Draw (C) and (d).

4- a) Prove that f admits an inverse function f^{-1} whose domain of definition is to be determined.

b) Draw the representative curve (C') of f^{-1} in the same system as (C).

c) Prove that (C') is tangent to $x'x$ and to (C) at two points to be determined.

5- a) Determine the point of inflection of (C').

b) Determine the point L of (C') where the tangent to (C') is parallel to (d).

c) Determine the coordinates of L in the direct orthonormal system $(O\,;\overrightarrow{u}\,,\overrightarrow{v})$ such that

$$\left\|\overrightarrow{u}\right\| = 1 \text{ and } (\overrightarrow{i}\,;\overrightarrow{u}) = 45^\circ .$$

B- Consider the integral $I_n = \displaystyle\int_p^e x(\ell n x)^n\,dx$ where $n \in IN$ and $p \in \,]0\,;e[$. Let $J_n = \lim\limits_{p\to 0^+} I_n$.

1- a) Using integration by parts, prove that, for all n in IN, $I_{n+1} = \dfrac{1}{2}\left[e^2 - p^2(\ell n\, p)^{n+1} - (n+1)I_n\right]$.

b) Calculate I_0. Deduce I_1 and I_2, then calculate J_0, J_1 and J_2.

2- a) Express the integral $J = \displaystyle\int_p^e f(x)\,dx$ in terms of I_0, I_1 and I_2 then calculate $\lim\limits_{p\to 0^+} J$.

b) Calculate the area of the domain bounded by (C), $x'x$, $y'y$ and the line of equation $x = e$.

c) Calculate the area S of the closed domain bounded by (C) and (C'). Verify that S is half the area of the square of diagonal $[OE]$ where $E(e\,;e)$.

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Solution of MATHEMATICS

Exercise 1

1- $f(1) = f(1\times 1) = f(1) + f(1) = 2f(1)$. Therefore $f(1) = 0$.

For all x in I, $f(\frac{1}{x}) + f(x) = f(x\times \frac{1}{x}) = f(1) = 0$. Therefore $f(\frac{1}{x}) = -f(x)$

2- For all a and b in I, $f(\frac{a}{b}) = f(a\times \frac{1}{b}) = f(a) + f(\frac{1}{b}) = f(a) - f(b)$.

3- a) For all a and b in I, If $a < b$ then $\frac{a}{b} \in]0\,;1[$ and $f(\frac{a}{b}) < 0$.

Consequently, $f(a) - f(b) < 0$ and $f(a) < f(b)$. Therefore, f is strictly increasing on I .

Inverse function

- f is continuous on I since f is differentiable on I .
- f is strictly increasing on I .

Therefore, f has an inverse function g defined on $f(I) = IR$.

b) For all a and b in IR, $g(a) \in I$ and $g(b) \in I$.

Therefore, $f\left(g(a)\times g(b)\right) = f\left(g(a)\right) + f\left(g(b)\right) = a + b$.

Consequently, $g(a)\times g(b) = f^{-1}(a+b) = g(a+b)$.

For all a in IR, $g(0) = g(a-a) = g(a)\times g(-a)$ where $g(0) = 1$ since $f(1) = 0$.

Finally, $g(a)\times g(-a) = 1$ and $g(-a) = \frac{1}{g(a)}$.

Exercise 2

1- a) If A is realized then the ball is drawn from the urn U that contains n balls of which 4 are red ; therefore, $p(R/A) = \frac{4}{n}$.

If B is realized then the ball is drawn from the urn V that contains n balls of which 1 is red ; therefore, $p(R/B) = \frac{1}{n}$.

If C is realized then the ball is drawn from the urn W that contains n balls of which 2 are red ; therefore, $p(R/C) = \frac{2}{n}$.

b) $p(R) = p(A)\times p(R/A) + p(B)\times p(R/B) + p(C)\times p(R/C)$.

$$=\frac{1}{6}\times\frac{4}{n}\ +\ \frac{1}{3}\times\frac{1}{n}+\ \frac{1}{2}\times\frac{2}{n}=\frac{2}{n}.$$

$p(R)=p(R/C)=\dfrac{2}{n}$; therefore the events C and R are independen

2- a) $p(A/R)=\dfrac{p(A\cap R)}{p(R)}=\dfrac{p(A)\times p(R/A)}{p(R)}=\dfrac{1}{6}\times\dfrac{4}{n}\ \div\ \dfrac{2}{n}=\dfrac{1}{3}.$

b) $p(C/W)=p(C/\overline{R})=\dfrac{p(C\cap\overline{R})}{1-p(R)}=\dfrac{p(C)-p(C\cap R)}{1-p(R)}=\left(\dfrac{1}{2}-\dfrac{1}{2}\times\dfrac{2}{n}\right)\div\left(1-\dfrac{2}{n}\right)=\dfrac{n-2}{2(n-2)}=\dfrac{1}{2}$.

Or The events C and R are independent then , the events C and $\overline{R}$ are independent .

Therefore $p(C/W)=p(C/\overline{R})=p(C)=\dfrac{1}{2}$.

3- a) $p(R)>0.4$; $\dfrac{2}{n}>0.4$; $n<5$. Therefore $n=4$ (since $n\in IN$ and $n\geq 4$)

b) $p(\overline{R})=2\,p(R)$; $p(R)=\dfrac{1}{3}$; $n=6$.

4- When $n=6$, $p(R)=\dfrac{1}{3}$ and $p(\overline{R})=\dfrac{2}{3}$.

The two events " at least one of the 10 drawn balls is red " and " the 10 drawn balls are white " are opposite . The required probability is $p=1-\left(\dfrac{2}{3}\right)^{10}$.

Exercise 3

1- Let R be the rotation of center I that transforms A into B .

a) $R(A)=B$ and $(\overrightarrow{IA}\ ;\ \overrightarrow{IB})=-\dfrac{\pi}{2}\quad(2\pi)$. Therefore the angle of R is $-\dfrac{\pi}{2}$.

b) Construction of D such that $IC=ID$ and $(\overrightarrow{IC}\ ;\ \overrightarrow{ID})=-\dfrac{\pi}{2}\quad(2\pi)$.

$R(A)=B$ and $R(C)=D$; therefore $(\overrightarrow{AC}\ ;\ \overrightarrow{BD})=-\dfrac{\pi}{2}\quad(2\pi)$.

2- Let R' be the rotation of angle $\dfrac{\pi}{2}$ that transforms A into D .

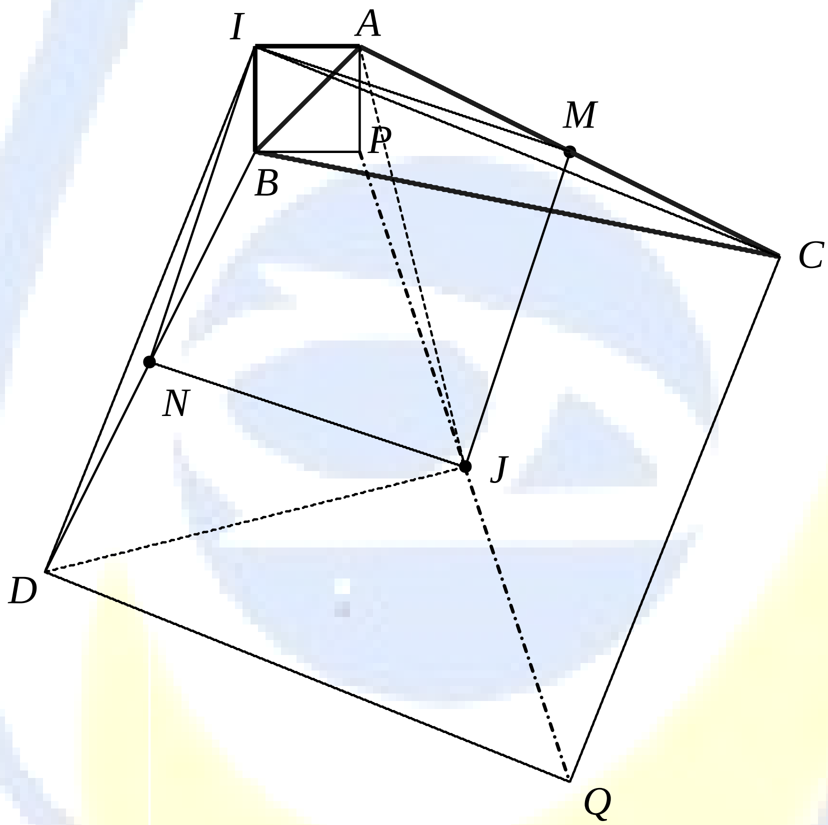
a) The center J of R' is such that $JA=JD$ and $(\overrightarrow{JA}\ ;\ \overrightarrow{JD})=\dfrac{\pi}{2}\quad(2\pi)$.

b) $(\overrightarrow{AC}\ ;\ \overrightarrow{DB})=(\overrightarrow{AC}\ ;\ \overrightarrow{BD})=\dfrac{\pi}{2}\quad(2\pi)$, $AC=DB$ and $R'(A)=D$. Therefore $R'(C)=B$.

3- Let M the mid point of $[AC]$ and N that of $[BD]$.

$R([AC]) = R'([AC]) = [BD]$; therefore $R(M) = R'(M) = N$.

Consequently , the triangles IMN and JMN are right and isosceles having the same hypotenuse $[MN]$. Therefore $IMJN$ is a square .



4-a) $IAPB$ is an indirect square . Therefore $IP = \sqrt{2}\, IA$ and $(\overrightarrow{IA}\ ;\ \overrightarrow{IP}) = -\frac{\pi}{4}\ \ (2\pi)$.

Then $S = Sim(I\ ;\ \sqrt{2}\ ;\ -\frac{\pi}{4})$.

b) $ICQD$ is an indirect square . Therefore $IQ = \sqrt{2}\, IC$ and $(\overrightarrow{IC}\ ;\ \overrightarrow{IQ}) = -\frac{\pi}{4}\ \ (2\pi)$.

Therefore $S(C) = Q$.

$IMJN$ is an indirect square . Therefore $IJ = \sqrt{2}\, IM$ and $(\overrightarrow{IM}\ ;\ \overrightarrow{IJ}) = -\frac{\pi}{4}\ \ (2\pi)$.

Therefore $S(M) = J$.
$S(A) = P$, $S(C) = Q$, $S(M) = J$ and M the mid point of $[AC]$.
Therefore J is the mid point of $[PQ]$.

5- The plane is referred to a direct orthonormal system $(O\,;\,\vec{u}\,,\,\vec{v})$ such that $z_A = 2 - i$, $z_B = 1 - 2i$ and $z_C = 6 - 3i$.

a) The complex relation of the rotation R of angle $-\frac{\pi}{2}$ is $z' = -\,i\,z + b$ such that $z_B = -\,i\,z_A + b$.
Hence $b = 2$ and $z' = -\,i\,z + 2$.
The affix of its center I of R is $z_I = \frac{2}{1+i} = 1 - i$.

b) $R(C) = D$; therefore $z_D = -\,i\,z_C + 2 = -1 - 6i$

c) The complex relation of the similitude $S(I\,;\,\sqrt{2}\,;\,-\frac{\pi}{4})$ is $z' = \sqrt{2}\,e^{-i\frac{\pi}{4}} z + (1 - \sqrt{2}\,e^{-i\frac{\pi}{4}})z_I$
$z' = (1 - i)\,z + i\,(1 - i)$; $z' = (1 - i)\,z + 1 + i$.

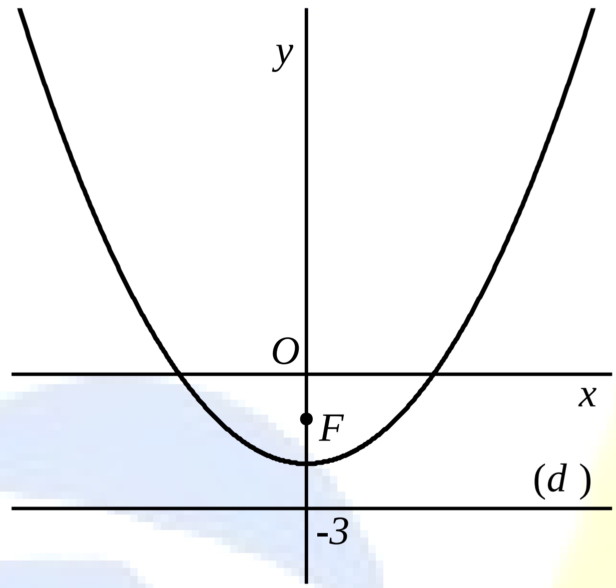
d) $S(M) = J$, then $z_J = (1 - i)\,z_M + 1 + i$ where $z_M = \frac{1}{2}(z_A + z_C) = 4 - 2\,i$; therefore $z_J = 3 - 5\,i$.
$S(A) = P$, then $z_P = (1 - i)\,z_A + 1 + i = 2 - 2\,i$.
$S(C) = Q$, then $z_Q = (1 - i)\,z_C + 1 + i = 4 - 8\,i$.
$z_J = \frac{1}{2}(z_P + z_Q)$. Therefore J is the midpoint of $[PQ]$.

Exercise 4

A- $M\,(4\cos\alpha\,;\,2\cos2\alpha)$, where α is a real number .
$2\,\cos2\alpha = 2(2\cos^2\alpha - 1) = 4\,\cos^2\alpha - 2$; then , as α varies , the point M varies on the parabola of equation $y = \frac{1}{4}x^2 - 2$.
As α traces IR , the abscissa of M traces the interval $[-4\,;\,4]$. Therefore the set of M is the part of the parabola of equation $y = \frac{1}{4}x^2 - 2$ that corresponds to $x \in [-4\,;\,4]$.

B- (P) is the parabola of equation $x^2 = 4(y+2)$.

1- The vertex of (P) is $(0\,;-2)$;
the focal axis is $y'y$ and the parameter is 2 .
Therefore :
the focus of (P) is $F(0\,;-1)$ and
the directrix is $(d): y=-3$.
Drawing (P).



2- $A(\alpha\,;\frac{1}{4}\alpha^2-2)$ and $B(\beta\,;\frac{1}{4}\beta^2-2)$.
(d_1) is the tangent to (P) at A and (d_2) is the tangent to (P) at B.

a) The slopes of (d_1) and (d_2) are respectively $\frac{\alpha}{2}$ and $\frac{\beta}{2}$.

$(d_1)\ :\ y=\frac{\alpha}{2}x-\frac{\alpha^2}{4}-2$ and

$(d_2)\ :\ y=\frac{\beta}{2}x-\frac{\beta^2}{4}-2$.

b) $(d_1)\cap(d_2)\ :\ 2(\alpha-\beta)x=\alpha^2-\beta^2\ ;\ x=\frac{\alpha+\beta}{2}$.

(d_1) and (d_2) intersect at the point $T\left(\frac{\alpha+\beta}{2}\,;\frac{\alpha\beta-8}{4}\right)$.

3- (d_1) and (d_2) are perpendicular if and only if $\frac{\alpha}{2}\times\frac{\beta}{2}=-1$; that is $\alpha\beta=-4$.

If $\alpha\beta=-4$, then $y_T=-3$ and T belongs to the directrix (d) of (P) .

$\overrightarrow{FA}\,(\alpha\,;\frac{\alpha^2}{4}-1)$ and $\overrightarrow{FB}\,(\beta\,;\frac{\beta^2}{4}-1)$.

$$Det(\overrightarrow{FA}\ ;\overrightarrow{FB})=\alpha\left(\frac{\beta^2}{4}-1\right)-\beta\left(\frac{\alpha^2}{4}-1\right)=\frac{\alpha\beta}{4}(\beta-\alpha)+\beta-\alpha\ .$$

If $\alpha\beta=-4$, then $Det(\overrightarrow{FA}\ ;\overrightarrow{FB})=0$ and (AB) passes through the focus F of (P)

4- (d_1) and (d_2) are not perpendicular and θ is a measure of their acute angle .

- $\overrightarrow{n_1}(\alpha\ ;-2)$ is a normal vector of (d_1) and $\overrightarrow{n_2}(\beta\ ;-2)$ is a normal vector of (d_2) ; then ,

$$\cos\theta = \frac{\left|\overrightarrow{n_1}\cdot\overrightarrow{n_2}\right|}{\left\|\overrightarrow{n_1}\right\|\times\left\|\overrightarrow{n_2}\right\|} = \frac{|\alpha\beta+4|}{\sqrt{(4+\alpha^2)(4+\beta^2)}} .$$

- The slopes of (d_1) and (d_2) are respectively $\frac{\alpha}{2}$ and $\frac{\beta}{2}$; then ,

$$\tan\theta = \left|\frac{\frac{\alpha}{2}-\frac{\beta}{2}}{1+\frac{\alpha}{2}\times\frac{\beta}{2}}\right| = 2\left|\frac{\alpha-\beta}{4+\alpha\beta}\right| .$$

5- a) If α and β vary such that $\theta = 45^\circ$, then $\cos\theta = \frac{\sqrt{2}}{2}$ (or $\tan\theta = 1$) .

Therefore , $4\alpha^2+4\beta^2-\alpha^2\beta^2-16\alpha\beta = 16$.

$4(\alpha+\beta)^2-(\alpha\beta+12)^2 = 4\alpha^2+4\beta^2-\alpha^2\beta^2-16\alpha\beta-144 = 16-144 = -128 = \text{constant}$.

The coordinates x and y of T are such that $\alpha+\beta = 2x$ and $\alpha\beta = 4y+8$. Therefore ,

If α and β vary such that $\theta = 45^\circ$, then $x^2-(y+5)^2 = -8$. Hence T varies on the hyperbola (H) of equation $(y+5)^2-x^2 = 8$.

b) The center of (H) is the point $(0\ ;-5)$;

the focal axis of (H) is the y-axis , $a = b = 2\sqrt{2}$ and $c = a\sqrt{2} = 4$; then :

a focus of (H) is the point $(0\ ;-5+c)$ which and the focus F of (P) ,

the corresponding directrix is the straight line of equation $y = -5+\frac{a^2}{c}$; $y = -3$ which is the directrix (d) of (P).

Exercise 5

A- The function f is defined on $[0\ ;+\infty[$ by $f(0) = 0$ and $f(x) = x(\ell n x)^2 - 2x\,\ell n x + 2x$ for $x \neq 0$.

1- a) $\lim\limits_{x\to 0^+} x\,\ell n x = \lim\limits_{x\to 0^+} x(\ell n x)^2 = 0$; therefore , $\lim\limits_{x\to 0^+} f(x) = 0 = f(0)$ and f is continuous at 0 .

b) $\lim\limits_{x\to 0^+}\frac{f(x)}{x} = \lim\limits_{x\to 0^+}\left[(\ell n x)^2 - 2\,\ell n x + 2\right] = +\infty$. Therefore , the tangent to (C) at $O(0\ ;0)$ is $y'y$.

2- $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left[x\,\ell n x(\ell n x - 2) + 2x\right] = +\infty$

For $x \neq 0$, $f'(x) = (\ell n\, x)^2$.

x	0		1		$+\infty$
$f'(x)$	‖	$+$	0	$+$	
$f(x)$	0	↗	2	↗	$+\infty$

3- $f(e) = e$ and $f'(e) = 1$; (d) : $y = x$.

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \left[\ell n\, x(\ell n\, x - 2) + 2\right] = +\infty \ .$$

(C) has , at $+\infty$, an asymptotic direction parallel to $y'y$
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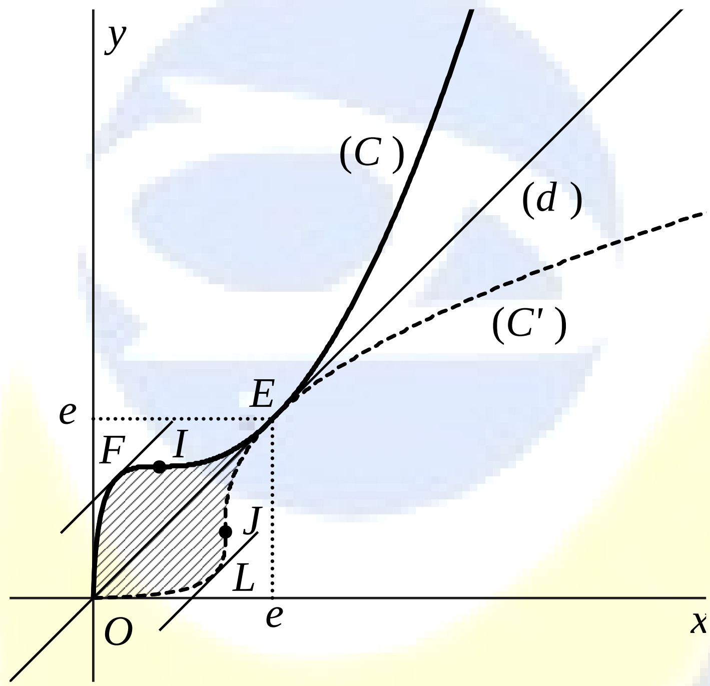
Drawing (C) and (d) .



4- a) f is continuous and strictly increasing ; it admits an inverse function f^{-1} defined on $f([0\,;+\infty[)$ which is $[0\,;+\infty[$.

b) (C') is the symmetric of (C) with respect to the straight line (d) of equation $y = x$.

c) (C) is tangent to $y'y$ at $O(0\,;0)$; then , by symmetry with respect to (d), (C') is tangent to $x'x$ at $O(0\,;0)$.

(C) is tangent to (d) at $E(e\,;e)$; then (C') is tangent to (d) at $E(e\,;e)$.

(C') and (C) have the same tangent at E ; they are tangent at this point .

5- a) For $x \neq 0$, $f''(x) = 2\dfrac{\ell n\, x}{x}$.

$f''(x)$ changes sign at 1 . Therefore (C) admits the point $I(1\,;\,2)$ as a point of inflection .
By symmetry with respect to the straight line (d), (C') admits the point $J(2\,;\,1)$ as a point of inflection .

b) $f'(x) = 1$ is equivalent to $\ell n\, x = 1$ or $\ell n\, x = -1$, that is $x = e$ or $x = \dfrac{1}{e}$.

The tangent to (C) at the point $F\left(\dfrac{1}{e}\,;\,\dfrac{5}{e}\right)$ is parallel to (d); then , by symmetry with respect to (d), the tangent to (C') at the point $L\left(\dfrac{5}{e}\,;\,\dfrac{1}{e}\right)$ is parallel to (d).

Or

The tangent to (C') at the point of abscissa x is parallel to (d) if and only if $x \neq e$ and $\left(f^{-1}\right)'(x) = 1$; But $\left(f^{-1}\right)'(x) \times f'\left(f^{-1}(x)\right) = 1$; therefore $f'\left(f^{-1}(x)\right) = 1$;

$\ell n\left(f^{-1}(x)\right) = 1$ or $\ell n\left(f^{-1}(x)\right) = -1$; $f^{-1}(x) = e$ or $f^{-1}(x) = \dfrac{1}{e}$;

$x = f(e) = e$ or $x = f(\dfrac{1}{e}) = \dfrac{5}{e}$.

Finally the tangent to (C') at the point $L\left(\dfrac{5}{e}\,;\,\dfrac{1}{e}\right)$ is parallel to (d).

c) The coordinates of L in the orthonormal system $(O\,;\,\overrightarrow{u}\,,\,\overrightarrow{v}\,)$ are x and y such that

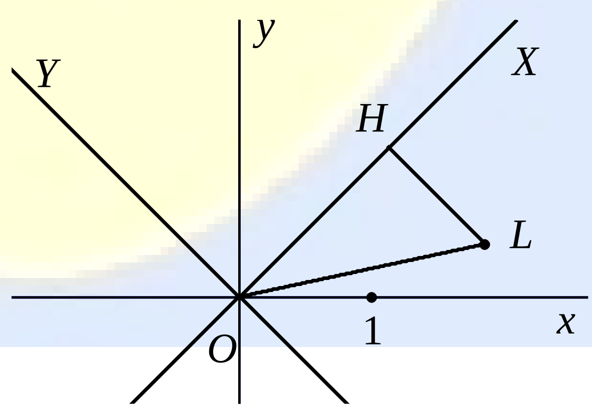
$$x = \frac{5}{e}\cos\left(\frac{-\pi}{4}\right) - \frac{1}{e}\sin\left(\frac{-\pi}{4}\right) = \frac{3\sqrt{2}}{e} \quad \text{and} \quad y = \frac{5}{e}\sin\left(\frac{-\pi}{4}\right) + \frac{1}{e}\cos\left(\frac{-\pi}{4}\right) = \frac{-2\sqrt{2}}{e} .$$

Or

The distance from L to (d) is $LH = \dfrac{2\sqrt{2}}{e}$;

therefore $y = \dfrac{-2\sqrt{2}}{e}$.

$x = OH = \sqrt{OL^2 - LH^2} = \dfrac{3\sqrt{2}}{e}$.



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B- $I_n = \int_p^e x(\ell n x)^n \, dx$ where n is a natural number and p is a real number belonging to $]0\,;e[$.

1- a) Let $U = (\ell n x)^{n+1}$ and $V' = x$ then, $U' = \frac{n+1}{x}(\ell n x)^n$ and $V = \frac{x^2}{2}$.

Using integration by parts ,

$$I_{n+1} = \left[\frac{x^2}{2}(\ell n x)^{n+1}\right]_p^e - \frac{n+1}{2}\int_p^e x(\ell n x)^n \, dx = \frac{1}{2}\left[e^2 - p^2(\ell n\, p)^{n+1} - (n+1)I_n\right].$$

b) $I_0 = \int_p^e x dx = \left[\frac{1}{2}x^2\right]_p^e = \frac{1}{2}(e^2 - p^2)$.

For $n = 0$, $I_1 = \frac{1}{2}\left[e^2 - p^2\,\ell n\, p - I_0\right] = \frac{1}{4}(e^2 + p^2) - \frac{1}{2}p^2\,\ell n\, p$;

for $n = 1$, $I_2 = \frac{1}{2}\left[e^2 - p^2(\ell n\, p)^2 - 2I_1\right] = \frac{1}{4}(e^2 - p^2) - \frac{1}{2}p^2(\ell n\, p)^2 + \frac{1}{2}p^2\,\ell n\, p$.

$J_0 = \lim_{p\to 0^+} I_0 = \lim_{p\to 0^+} \frac{1}{2}(e^2 - p^2) = \frac{1}{2}e^2$.

$J_1 = \lim_{p\to 0^+} I_1 = \frac{1}{4}e^2$.

$J_2 = \lim_{p\to 0^+} I_2 = \frac{1}{4}e^2$.

2- a) $J = \int_p^e f(x)\,dx = I_2 - 2I_1 + 2I_0$; $\lim_{p\to 0^+} J = J_2 - 2J_1 + 2J_0 = \frac{3}{4}e^2$

b) The area of the domain bounded by (C), $x'x$, $y'y$ and the straight line of equation $x = e$

is $A = \lim_{p\to 0^+} J$ units of area ; $A = \frac{3}{4}e^2$ units of area .

c) By symmetry with respect to the straight line (d), $S = 2S_1$ where S_1 is the area of the domain bounded by (C), (d), $y'y$ and the straight line of equation $x = e$.

$S = 2[A - A_1]$ where A_1 is the area of the triangle OEF ; $A_1 = \frac{1}{2}e^2$ units of area .

Therefore $S = 2\left(\frac{3}{4}e^2 - \frac{1}{2}e^2\right) = \frac{1}{2}e^2$ units of area $= \frac{1}{2}$ area of the square of diagonal $[OE]$.