LEBANESE UNIVERSITY
FACULTY OF ENGINEERING

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Entrance Exam 2007-2008 **Duration: 3 hours**

MATHEMATICS

I-We are given an urn containing 5 red balls and 3 black balls and a weighted coin so that the probabilities of tails and heads are proportional to 2 and 3 .

A player tosses the coin.

If the coin turns up tails, the player draws at random 2 balls from the urn .

If the coin turns up heads, the player draws at random 3 balls from the urn .

The player wins the game if all balls drawn from the urn are red .

Consider the events:

H : " the coin turns up a head " ; ***T*** : " the coin turns up a tail " and ***W*** : " the player wins the game " .

1- Prove that the probability $p(H)$ of the event H is equal to 0.6 and calculate $p(T)$.

2- Calculate $p(W/H)$, $p(W/T)$ and $p(W)$.

3- Calculate the probability of each of the following events :

A : " the coin turns up heads knowing that the player wins the game ".

B : " the coin turns up tails knowing that the player loses the game ".

4- Suppose in this part that the player repeat the game n times by replacing the drawn balls in the urn after each drawing .

a) Calculate , in terms of n, the probability p that he wins the game at least once .

b) Determine the least value of the natural number n so that $p > 0.94$.

II- In the adjacent figure $ABCD$ and $ALPE$ are two direct squares.

Consider the rotation $R = r(A\,;\frac{\pi}{2})$, the similitude $S = s(D\,;\sqrt{2}\,;-\frac{\pi}{4})$ and their composition $f = R \circ S$.

1- Let O be the center of the square $ABCD$.

a) Determine $f(C)$ and $f(O)$. Deduce the point $A' = f(A)$.

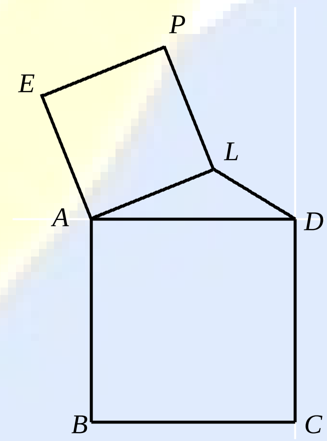
b) Prove that f is a similitude of center B whose ratio and angle are to be determined .

c) Let $L' = f(L)$. Prove that $\overrightarrow{A'L'} = \overrightarrow{AP}$ and plot L'.

2- Let $g = \underbrace{f \circ f \circ f \circ \cdots\cdots \circ f}_{n\ times}$.

Determine the natural number $n > 2$ so that g is a dilation (homothetie).

Determine the ratio of this dilation according to the values of n.



In what follows , refer the plane to the direct orthonormal system $(A\,;\vec{u}\,,\vec{v})$ *such that* $D(5\,;0)$ *and suppose that* $L(3\,;2)$.

3- a) Determine the affixes z_1 and z_2 of B and E respectively .

b) Calculate the affix z_3 of the mid point H of $[DL]$ and prove that $\frac{z_1 - z_2}{z_3}$ is pure imaginary.

c) Deduce that the median relative to $[DL]$ in triangle ADL is a height in triangle ABE .

4- a) Determine the complex relation of the similitude f .

b) Determine the affix of L' and verify that $\overrightarrow{A'L'} = \overrightarrow{AP}$.

III- *A*- Let (H) be the hyperbola of equation $x^2 - y^2 = 4$.

1- a) Determine the vertices and the asymptotes of (H). Construct (H).

b) Determine the foci F' and F $(x_F > 0)$ and the corresponding directrices (d') and (d) of (H).

2- Let $P(x_0 \ ; \ y_0)$ be any point of (H) and (T) the tangent line at P to (H) .
(T) cuts the asymptotes of (H) at R and S . Prove that P is the mid point of $[RS]$.

***B*-** Let (S) be the set of points of the plane deprived from the origin O.
Designate by f the mapping that associates to any point $M(\alpha \ ; \ \beta)$ of (S) the straight line (Δ) of equation $\alpha x - \beta y = 4$.

1- Let (Δ) and (Δ') be the straight lines associated to two distinct points M and M' of (S).
Prove that if (Δ) passes through M' then (Δ') passes through M .

2- a) Prove that the straight line associated to the focus F is the directrix (d) of (H).

b) Prove that if M belongs to (d), then (Δ) passes through F and is perpendicular to (MF).

c) Prove that if M belongs to (H), then (Δ) is the tangent to (H) at M .

3- Let M be a given point on the directrix (d) of (H).

a) Construct , on the drawn figure , the straight line (Δ) associated to M .

b) Describe a geometric construction of the tangents to (H) drawn through M .

IV- *A*- Let f be the function defined on $]0 \ ; +\infty[$ by $f(x) = \frac{2\ell n\, x}{x^2} - \frac{1}{x^2} + 1$.

1- a) Calculate the limits of f at the bounds of its domain of definition .

b) Prove that $f'(x) = \frac{4(1 - \ell n\, x)}{x^3}$.

2- a) Set up the table of variation of f.

b) Calculate $f(1)$ and determine the sign of $f(x)$.

c) Calculate $f(\sqrt{e})$. Deduce that :

- *For all* $t \in [\sqrt{e} \ ; +\infty[\ , \ f(t) \geq 1$ (1)
- *For all* $t \in]0 \ ; \sqrt{e}\ ] \ , \ f(t) \leq 1$ (2)

B- Let g and h be the functions defined on $]0\,;+\infty[$ by $g(x)=\int_{\sqrt{e}}^{x} f(t)\,dt$ and $h(x)=\frac{e-2}{\sqrt{e}}+g(x)$.

1- a) Using the inequalities (1) and (2) proved in ***A***-2-c) , prove that :
For all x in the interval $]0\,;+\infty[$, $g(x)\ \geq\ x-\sqrt{e}$.

b) Deduce that for all x in the interval $]0\,;+\infty[$, $h(x)\ \geq\ x-\frac{2}{\sqrt{e}}$.

2- a) Using integration by parts , calculate $\int_{\sqrt{e}}^{x}\frac{\ell n\,t}{t^2}dt$ where $x>0$.

b) Deduce $\int_{\sqrt{e}}^{x} f(t)\,dt$ and prove that $h(x)=x-\frac{1}{x}-\frac{2\ell n\,x}{x}$.

3- Let (C) be the representative curve of h in an orthonormal system $(O\ ;\vec{i}\ ,\ \vec{j}\ )$. (**Unit : 2 cm**) .

a) Determine an equation of the tangent (d) to (C) at the point of abscissa $\sqrt{e}$.

b) Using the preceding parts , determine the relative position of (d) and (C).

4- a) Calculate $\lim\limits_{x\to+\infty}[h(x)-x]$. Deduce that (C) has an oblique asymptote (D) at $+\infty$.

b) Determine the point of intersection of (C) and (D).

5- a) Set up the table of variation of h.

b) Construct (D), (d) and (C).

c) Calculate the area of the domain bounded by (C), (D) and the lines of equation $x=1$ and $x=e$.

6- a) Prove that the restriction of h to the interval $[1\,;+\infty[$ admits an inverse function h^{-1}.

b) The curve (C') of h^{-1} admits a tangent (d') parallel to (d) . Determine their point of contact and draw (d') and (C') in the same system as (C).

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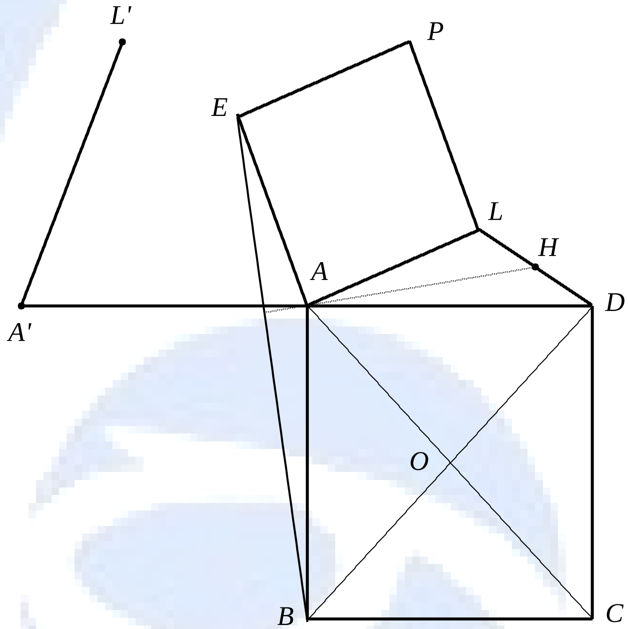
Solution of Mathematics

Exercice 1

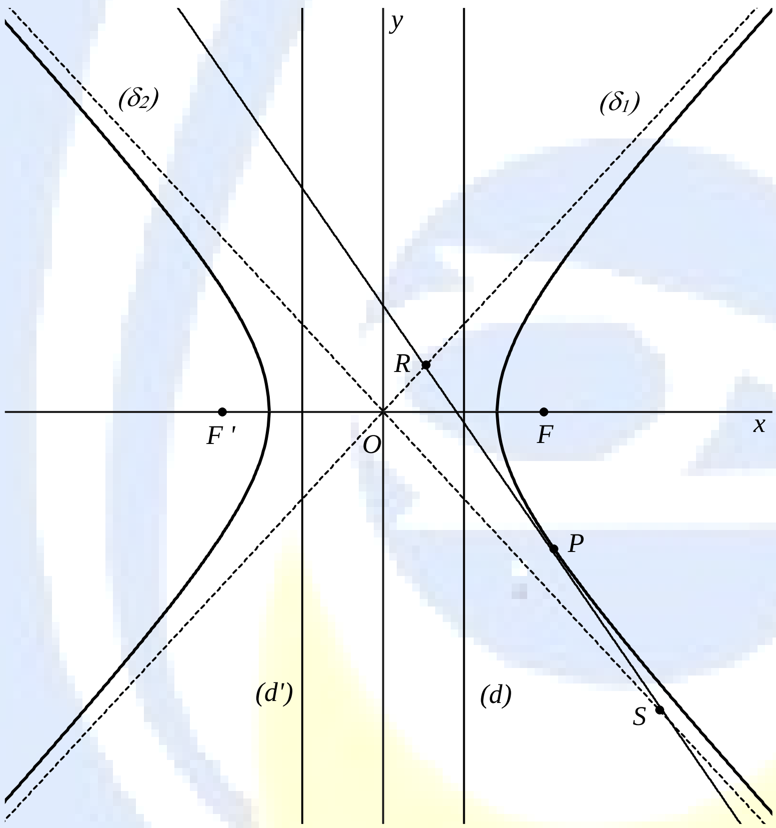
	Elements of answers	Notes
1	$p(F)$ and $p(P)$ are such that $p(F)=3k$, $p(P)=2k$ and $p(F)+p(P)=1$. where $k=\frac{1}{5}$, $p(F)=\frac{3}{5}$ and $p(P)=\frac{2}{5}$.	
2	If F is realized, the player draws at random 3 balls from the urn that contains 8 balls of which 5 are red. Therefore, $p(G/F)=\frac{C_5^3}{C_8^3}=\frac{5}{28}$. $p(G/P)=\frac{C_5^2}{C_8^2}=\frac{5}{14}$. $p(G)=p(F)\times p(G/F)+p(P)\times p(G/P)=\frac{3}{5}\times\frac{5}{28}+\frac{2}{5}\times\frac{5}{14}=\frac{1}{4}$	
3	$p(A)=p(F/G)=\frac{p(G\cap F)}{p(G)}=\frac{p(F)\times p(G/F)}{p(G)}=\frac{3}{7}$. $p(B)=p(P/\overline{G})=\frac{p(\overline{G}\cap P)}{p(\overline{G})}=\frac{p(P)-p(G\cap P)}{p(\overline{G})}=\frac{\frac{2}{5}-\frac{1}{7}}{1-\frac{1}{4}}=\frac{12}{35}$.	
4a	$p=1-\left(\frac{3}{4}\right)^n$.	
4b	$p>0,94$; $\left(\frac{3}{4}\right)^n<0,06$; $n>9,77$. The smallest value of n such that $p>0,94$ is $n=10$.	

Exercice 2

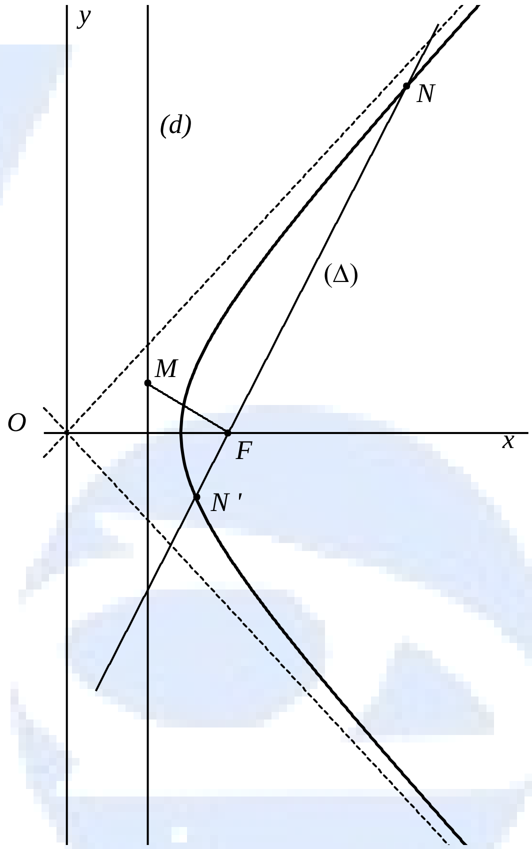
	Elements of answers	Notes
1a	$f(C) = R \circ S(C) = R(S(C))$ $\quad = R(B) = D$. $f(O) = R \circ S(O) = R(S(O))$ $\quad = R(A) = A$. All similitude preserves the mid-point. O is the mid-point of $[CA]$ then, A is the mid-point de $[DA']$. A' is the symmetric of D with respect to A . 	
1b	$f = R \circ S = Sim(A\,;1\,;\frac{\pi}{2}) \circ Sim(D\,;\sqrt{2}\,;-\frac{\pi}{4}) = Sim(......;\sqrt{2}\,;\frac{\pi}{4})$. $f(C) = D$; $BD = \sqrt{2}\,BC$ and $(\overrightarrow{BC}\,;\overrightarrow{BD}) = \frac{\pi}{4} \quad (2\pi)$, then, B is the center of f .	
1c	$f(L) = L'$ and $f(A) = A'$, then, $A'L' = \sqrt{2}\,AL$ et $(\overrightarrow{AL}\,;\overrightarrow{A'L'}) = \frac{\pi}{4} \quad (2\pi)$. Or $AP = \sqrt{2}\,AL$, $(\overrightarrow{AL}\,;\overrightarrow{AP}) = \frac{\pi}{4} \quad (2\pi)$. Therefore, $\overrightarrow{A'L'} = \overrightarrow{AP}$.	
2	$f = Sim(B\,;\sqrt{2}\,;\frac{\pi}{4})$ then, $g = Sim(B\,;(\sqrt{2})^n\,;n\frac{\pi}{4})$. g is a homothetic if and only if, $n\frac{\pi}{4} = k\pi$. where $n = 4k$ with $k \in IN^*$. The rapport of g is $\begin{cases} (\sqrt{2})^n & if\ k\ is\, even \\ -(\sqrt{2})^n & if\ k\ is\, odd \end{cases}$.	
3a	$z_A = 0$; $z_D = 5$; $z_B = -5i$; $z_L = 3 + 2i$ and $z_E = i\,z_L = -2 + 3i$	
3b	Let H be the mid-point of $[DL]$; $z_H = \frac{z_D + z_L}{2} = 4 + i$.	

	$\frac{z_1 - z_2}{z_3} = \frac{z_B - z_E}{z_H} = \frac{2-8i}{4+i} = -2i$ which is a pure imaginary.	
3c	$\frac{z_1 - z_2}{z_3} = \frac{z_B - z_E}{z_H - z_A}$. Therefore, $\frac{z_B - z_E}{z_H - z_A}$ is a pure imaginary. Hence, $(AH) \perp (BE)$. The relative median to $[DL]$ of triangle ADL is a height of triangle ABE .	
4a	The complex relation of f is $z' = (1+i)z - 5$.	
4b	$L' = f(L)$ and $z_L = 3+2i$, then, $z_L' = -4+5i$. $A' = f(A)$ and $z_A = 0$, then $z_A' = -5$. $z_P = z_L + z_E = 1+5i$ then, $z_L' - z_A' = 1+5i = z_P - z_A$. Hence, $\overrightarrow{A'L'} = \overrightarrow{AP}$	

Exercice 3

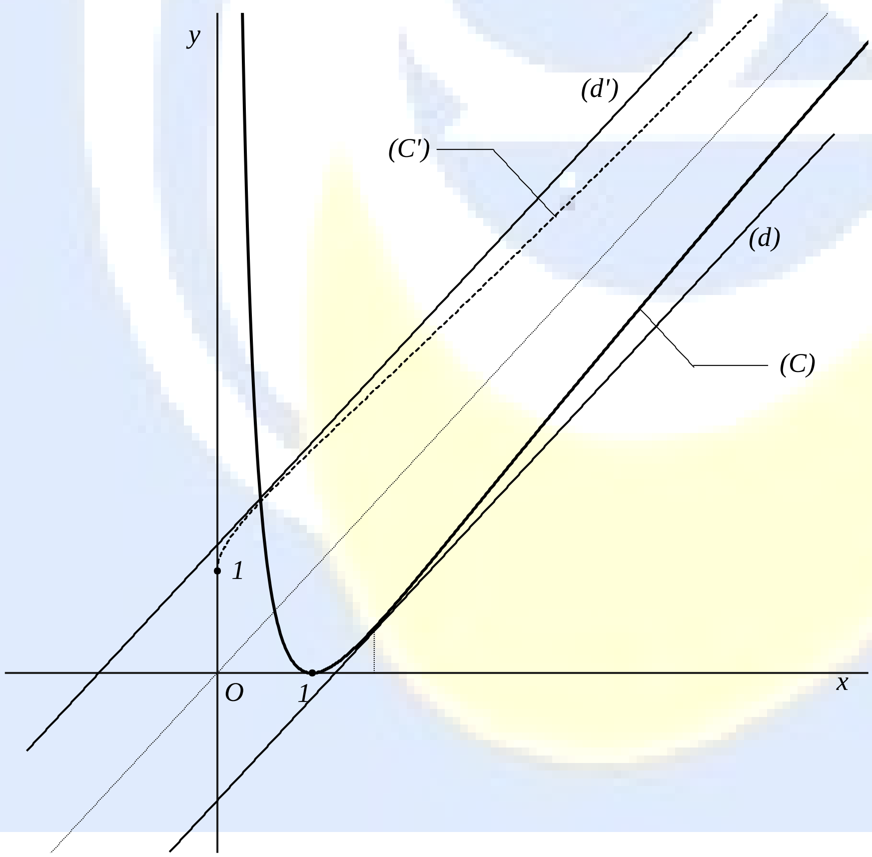
	Elements of answers	Notes
A1a	$a=2$ and the focal axis of (H) is $x'x$; the vertices of (H) are $A(2;0)$ and $A'(-2;0)$. The asymptote of (H) are (δ_1) : $y=x$ and (δ_2) : $y=-x$ 	
A1b	$c=a\sqrt{2}=2\sqrt{2}$; The focus of (H) are $F(2\sqrt{2}\ ;0)$ and $F'(-2\sqrt{2}\ ;0)$. $\frac{a^2}{c}=\sqrt{2}$; The directory of (H) are $(d): x=\sqrt{2}$ and $(d'): x=-\sqrt{2}$.	
A2	$P(x_0\ ;\ y_0)\in(H)$; $(T)\ :\ x_0x-y_0y=4$; (T) cuts (δ) at R and (δ') at S such that $x_R=\frac{4}{x_0-y_0}$ and $x_S=\frac{4}{x_0+y_0}$. $x_R+x_S=\frac{8x_0}{x_0^2-y_0^2}=\frac{8x_0}{4}=2x_0$.	

	P , R et S are aligned, then P is the mid-point of $[RS]$.	
B1	The straight line associated at point $M(\alpha\ ;\ \beta)$ is $(\Delta):\ \alpha x - \beta y = 4$; The straight line associated at point $M'(\alpha'\ ;\ \beta')$ is $(\Delta'):\ \alpha' x - \beta' y = 4$. If (Δ) passes through M' then, $\alpha\alpha' - \beta\beta' = 4$, which that (Δ') passes through M .	
B2a	The straight line associated at focus $F(2\sqrt{2}\ ;\ 0)$ by equation $2\sqrt{2}\,x = 4$ where $x = \sqrt{2}$ which is the directrix (d) associated at focus F .	
B2b	If $M(\sqrt{2}\ ;\ \beta)$, (Δ) : $\sqrt{2}\,x - \beta y = 4$; (Δ) passes through F . $\overrightarrow{MF}(\sqrt{2}\ ;\ -\beta)$ is a normal vector of (Δ) ; then, (Δ) is a perpendicular at (MF).	
B2c	If $M(\alpha\ ;\ \beta)$ belong to (H), the equation of the tangent in M at (H) is $\alpha x - \beta y = 4$ which is the equation of the straight line (Δ) associated at M .	

B3a	From B2b , the straight line (Δ) associated at point M of (d) is the perpendicular in F at (MF) . 	
B3b	The straight line (Δ) associated at point M of (d) cut the hyperbolic (H) in N and N'. the straight line associated at point N of (H) is the tangent (T) in N at (H) (from B2b) In addition, as the straight line (Δ) associated at point M passes through N then, the straight line (T) associated at point N passes through M (from B1) . (MN) is a tangent at (H) passes through M and , the same , (MN') is a tangent at (H) passes through M . By construction the tangents at (H) issues of M : By attached M at F , draw the perpendicular in F at (MF) cut the hyperbolic (H) in two point N and N'. The straight lines (MN) and (MN') are the tangents at (H) issues of M .	

Exercice 4

	Elements of answers	Notes
A1a	$\lim\limits_{x\to 0^+} f(x) = -\infty$ and $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left[1-\frac{1}{x^2}+2\frac{\ell n\, x}{x^2}\right]=1$	
A1b	$f'(x)=\frac{4(1-\ell n\, x)}{x^3}$.	
A2a	x : 0 ; 1 ; e ; + 😐 $f'(x)$: ‖ + 0 − $f(x)$: ‖ − 😐 ↗ 0 ↗ 1/e ² +1 ↘ 1	
A2b	$f(1)=0$; $f(x)<0$ for $x\in]0\,;1[$ and $f(x)>0$ for $x\in]1\,;+\infty[$	
A2c	$f(\sqrt{e})=1$; If $t\in[\sqrt{e}\ ;+\infty[$, $f(t)\in f([\sqrt{e}\ ;+\infty[)=[1\,;1+e^{-2}]$. If $t\in]0\,;\sqrt{e}]$, $f(t)\in f([]0\,;\sqrt{e}])=]-\infty\,;1]$.	
B1a	If $x\in[\sqrt{e}\ ;+\infty[$, then, $t\in[\sqrt{e}\ ;+\infty[$ *and* $f(t)\geq 1$. Where, $\int_{\sqrt{e}}^{x} f(t)\,dt \geq \int_{\sqrt{e}}^{x} dt = x-\sqrt{e}$ If $x\in]0\,;\ \sqrt{e}]$, then, $t\in]0\,;\ \sqrt{e}]$ *and* $f(t)\leq 1$. where, $\int_{\sqrt{e}}^{x} f(t)\,dt \geq \int_{\sqrt{e}}^{x} dt = x-\sqrt{e}$	
B1b	For all, x of interval $]0\,;+\infty[$, $g(x)\geq x-\sqrt{e}$ then, $h(x)\geq x-\frac{2}{\sqrt{e}}$.	
B2a	$\int_{\sqrt{e}}^{x}\frac{\ell n\, t}{t^2}dt=\frac{3}{2\sqrt{e}}-\frac{1+\ell n\, x}{x}$	
B2b	$\int_{\sqrt{e}}^{x} f(t)\,dt=\int_{\sqrt{e}}^{x}\left(\frac{2\ell n\, t}{t^2}-\frac{1}{t^2}+1\right)dt=2\int_{\sqrt{e}}^{x}\frac{\ell n\, t}{t^2}dt-\int_{\sqrt{e}}^{x}\frac{dt}{t^2}+\int_{\sqrt{e}}^{x}dt=\frac{2-e}{\sqrt{e}}+x-\frac{1}{x}-2\frac{\ell n\, x}{x}$. $h(x)=\frac{e-2}{\sqrt{e}}+\int_{\sqrt{e}}^{x} f(t)\,dt=x-\frac{1}{x}-\frac{2\ell n\, x}{x}$.	

<table>
<tr><td>B3a</td><td>$(d)\ :\ y = x - \frac{2}{\sqrt{e}}$.</td><td></td></tr>
<tr><td>B3b</td><td>For all, $x \in]0\,;+\infty[$, $h(x) \geq x - \frac{2}{\sqrt{e}}$ then, (C) is above (d) .</td><td></td></tr>
<tr><td>B4a</td><td>$\lim\limits_{x\to+\infty}[h(x)-x]=0$. The straight line (D) of equation $y = x$ is asymptotic at (C) at $+\infty$</td><td></td></tr>
<tr><td>B4b</td><td>$h(x) = x$ equivalent at $\ell n\, x = -\frac{1}{2}$. then, $(C)\cap(D)=\left\{(\frac{1}{\sqrt{e}}\,;\,\frac{1}{\sqrt{e}})\right\}$</td><td></td></tr>
<tr><td>B5a</td><td>$h'(x) = f(x)$
<table><tr><td>x</td><td>0</td><td></td><td>1</td><td></td><td>+ 😐</td></tr><tr><td>$h'(x)$</td><td></td><td>−</td><td>0</td><td>+</td><td></td></tr><tr><td>$h(x)$</td><td>+ 😐</td><td>↘</td><td>0</td><td>↗</td><td>+ 😐</td></tr></table></td><td></td></tr>
<tr><td>B5b</td><td></td><td></td></tr>
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B5c	$S=\int_1^e (x-h(x))dx=\int_1^e (\frac{1}{x}+2\frac{\ell n\,x}{x})dx=\left[\ell n\,x+\ell n^2 x\right]_1^e=2\ unit of\ area=8\ cm^2$.	
B6a	h is continuous and strictly increasing on $[1\,;+\infty[$; Then, h admits an inverse function h^{-1} defined on $[0\,;+\infty[$.	
B6b	(C') is the part of image (C) correspond to $x\in[1\,;+\infty[$ by symmetry with respect to (D) . The tangent (d') parallel at (d) is the image of (d) by symmetry with respect to (D) . the point of contact of (d) and (C) is $(\sqrt{e}\,;\sqrt{e}-\frac{2}{\sqrt{e}})$, (d') and (C') is $(\sqrt{e}-\frac{2}{\sqrt{e}}\,;\sqrt{e})$	